A Preferred-Habitat Model with a Corporate Sector[1]

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Abstract

I study the interplay of interest rate risk, credit risk, and bond quantities in a term structure model of Treasury and corporate bond yields. The core of the theory is an endogenous connection between credit and duration risk premia through bond portfolios. Shocks to default probabilities propagate to Treasury yields through their impact on the price of interest rate risk. The dependence of credit risk premia on interest rates affects the strength of monetary policy transmission to both long term Treasury and corporate yields. The credit and the duration risk premia amplify the effect of an increase in default rates on credit spreads. A decline in Treasury supply can adversely impact corporate yields by raising the price of credit risk through a safety channel. The impact of quantitative easing is asymmetric and depends on which assets are purchased.

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1 Introduction

The prices of Treasury and corporate bonds are tightly connected. Prices typically fall when interest rates rise, meaning that Treasury and corporate bonds both carry interest rate risk. When markets are integrated, absence of arbitrage implies that investors demand an identical compensation for exposure to the same risk factor. Further, Treasury bonds often serve as a benchmark for pricing corporate bonds, both in theory (Duffie & Singleton, 2003) and practice (PIMCO, 2024).

Despite this natural connection, however, the recent literature on term structure models has largely examined each market in isolation, and a unified account of the corporate and Treasury term structures is missing. Studying Treasury and corporate bonds jointly rather than separately is important to understand (i) how monetary policy and fundamental shocks propagate across markets[1], (ii) how investors' sensitivity to credit risk – e.g. during a flight to safety – affects Treasury term premia[2], (iii) why credit spreads move so much while default probabilities and fundamentals are less volatile[3], and (iv) how movements in the supply of each asset affect expected returns on the other asset[4]. Several empirical studies emphasize movements in bond risk premia, rather than default probabilities, to understand these broad phenomena (Du et al., 2019; Nozawa, 2017).

In this paper, I fill this gap by exploring, theoretically and empirically, the relation between Treasury and corporate bond risk premia[5]. My contribution is a novel economic mechanism that links the prices of interest rate and credit risk through bond portfolios. I deliver the key insights in a no-arbitrage model of the term structure of Treasury and corporate yields that formalizes the interplay of credit risk, duration risk, and bond quantities. Building on the preferred-habitat tradition (Gourinchas, Ray, & Vayanos, 2022; Vayanos & Vila, 2021), I introduce market segmentation through bond clienteles into an affine term structure of safe Treasury bonds and defaultable corporate bonds.

In the model, optimizing arbitrageurs, who are active in both markets, trade against a price-elastic habitat demand for Treasury and corporate bonds. Treasury bonds are default-free, whereas corporate bonds may default with positive probability. I assume that credit risk is fully diversifiable, but that fundamentals – i.e. default probabilities – vary over time. The key economic assumption is that Treasury and corporate bonds are in non-zero net supply. Because in equilibrium arbitrageurs carry

[1]Expansionary monetary policy lowers the term premium on government bonds and the external finance premium on risky corporate debt, see Bernanke and Kuttner (2005); Gertler and Karadi (2015); Gilchrist, López-Salido, and Zakrajšek (2015); Gilchrist and Zakrajšek (2012); Hanson and Stein (2015).

[2]See Adrian, Crump, and Vogt (2019) on flight to safety and Becker and Ivashina (2015) or Daniel, Garlappi, and Xiao (2021) on reaching for yield.

[3]See the literature on the credit spread puzzle and variation in bond discount rates, among others H. Chen (2010); L. Chen, Collin-Dufresne, and Goldstein (2008); Collin-Dufresne, Goldstein, and Martin (2001); Du, Elkamhi, and Ericsson (2019); Friewald and Nagler (2019); He, Khorrami, and Song (2022); Nozawa (2017).

[4]See the literature on supply effects in bond markets, e.g. D'Amico and King (2013); Greenwood and Vayanos (2014); Krishnamurthy and Vissing-Jorgensen (2012); Selgrad (2023).

[5]Recently, van Binsbergen, Nozawa, and Schwert (2023) emphasize the importance of evaluating corporate bond returns in excess of duration-matched Treasury returns.

exposure to interest rate and credit risk, their portfolios determine the price of duration and credit risk jointly. I model habitat demand through bond clienteles that only hold positions within their habitat. In addition to prices, I allow habitat demand to respond to default probabilities and interest rates directly. The different sensitivity of habitat demand to monetary policy and default probabilities is key to generate realistic movements in risk premia over time. Although this specification of habitat demand is stylized, it brings demand system asset pricing (Koijen & Yogo, 2019) into the class of affine term structure models (Dai & Singleton, 2000; Duffie & Kan, 1996).

The core of the theory is an *endogenous* connection between the credit and the duration risk premium through arbitrageurs' portfolios. In the equilibrium that I fully characterize, Treasury and corporate yields are affine in all the risk factors, including asset specific ones[6].
Market clearing conditions pin down risk prices, so that monetary policy and fundamentals impact credit and interest rate risk premia through their effect on residual supply. Since the latter responds to the short rate and to default probabilities, the price of credit and interest rate risk endogenously varies with interest rates and default probabilities. This result links the Treasury term premium to the excess bond premium and introduces a new channel through which monetary policy may affect the price of credit risk (Gertler & Karadi, 2015; Gilchrist & Zakrajšek, 2012).

I first derive implications for the transmission of monetary policy and fundamental shocks across markets. Treasury yields respond to changes in default probabilities, even when the risk factors are independent. This occurs because the price of interest rate risk endogenously varies with default rates. Because all bonds carry duration risk, fluctuations in the residual supply of corporate bonds driven by credit risk also impact arbitrageurs' exposure to interest rate risk. The subsequent change in the price of interest rate risk then affects Treasury yields. Treasuries load positively (negatively) on default probabilities when a deterioration in the credit quality raises (lowers) the market price of interest rate risk. The sign and the magnitude of the effect depend on habitat demand's sensitivity to credit risk, on arbitrageurs' risk aversion, and on default uncertainty.
Building on this mechanism, I provide a theoretical explanation for why transmission to long term rates is heterogeneous across assets, and how that depends on default uncertainty. Corporate yields overreact (underreact) to monetary policy shocks vis-à-vis Treasury yields when an increase in the short rate raises (lowers) the market price of credit risk. If Treasuries provide insurance against default risk, transmission to long term Treasury yields might be stronger when default uncertainty is high.

The second set of results characterizes the impact of asset substitutability and investors' appetite for credit risk on credit spreads. Credit spreads overreact (underreact) to changes in default probabilities when a deterioration in credit quality raises (lowers) the market price of credit risk. The effect is stronger when habitat investors aggressively substitute corporate bonds with Treasury bonds in response to an increase in credit risk, for example during a flight to safety. Monetary policy also moves credit spreads, but the effect is ambiguous. A contractionary monetary policy shock raises the price of interest rate risk, which typically has a greater effect on Treasury bonds. However, an

[6]Duration and credit risk factors to explain the cross-section of bond returns appear in Fama and French (1993).

increase in the short rate also moves the price of credit risk, so that the aggregate effect could go either way. These insights are useful to interpret, through the lens of the model, the *ex-ante* implications of a flight to safety and reaching for yield episodes on bond risk premia and credit spreads.

The third result shows that the credit and the interest rate risk premia depend on the aggregate quantities of both Treasury and corporate bonds. A positive supply shock in the corporate bond market increases arbitrageurs' exposure to interest rate and credit risk. As this affects the market price of duration and credit risk, the shock propagates to the Treasury market. This channel has implications for Quantitative Easing/Tightening (QE/QT) programs. If Treasury bonds have longer duration than corporate bonds, Treasury-only QE purchases may raise credit spreads. Another key message is that QE interventions can adversely impact corporate yields by raising the price of credit risk. Intuitively, arbitrageurs may want to hold Treasuries in order to hedge against deteriorating fundamentals. A decline in Treasury supply makes safe bonds scarcer and reduces the availability of insurance against fundamental shocks. These mechanisms are consistent with a safety and a portfolio rebalancing channel of QE, for which Selgrad (2023) provides empirical support.

I also discuss two broader implications for affine models of the term structure. First, when residual supply varies with interest rates and default probabilities, the risk prices of interest rate and credit risk are state-dependent. This result does not rely on any restriction on the dynamics of the risk factors, and it even holds in the special case that default rates and interest rates are independent. This has implications for the martingale dynamics of default intensity and interest risk, which is a key ingredient in reduced-form models of credit risk valuation. Second, unlike in standard CIR-style models (e.g. Duffee (2002) or Dai and Singleton (2000)), movements in risk premia need not be proportional to the volatility of the shocks. Indeed, my model generates time varying risk premia in a homoscedastic environment while preserving the tractability of affine term structure models[7].

To illustrate the model mechanisms, I calibrate its parameters by targeting the level and the volatility of the Treasury yield curve only. The model provides a good fit for both the corporate and the Treasury yield curve, especially at short to intermediate maturities. Although the implied volatility of credit spreads is higher than in the data, the implied credit spreads match the average credit spreads for BBB-rated issuers with a default intensity set to approximate historical default rates. The model captures the fact that the term structure of credit spreads is upward sloping for investment-grade issuers, but downward sloping for high-yield bonds (Sarig & Warga, 1989).

I then analyze the impact of monetary policy interventions on credit spreads and corporate bond yields. The first observation in my calibration is that default uncertainty weakens the transmission of monetary policy shocks throughout the term structure of corporate bonds. Yet, the opposite holds for the Treasury yield curve when Treasury yields load negatively on credit risk factors. The underreaction of forward rates to monetary policy shocks is less severe when default uncertainty

[7]Affine term structure models are tractable, but typically they cannot generate variation in risk premia that is independent the volatility of interest rate and the default intensity (Duffee, 2002).

is high. As predicted by the theoretical analysis, higher default uncertainty increases the value of Treasuries as hedges against default risk, lowering the risk premia on government bonds. When analyzing QE interventions, the model captures spillover effects across markets. The impact of QE on bond yields is asymmetric and depends on which assets are purchased. In the calibration, the response of credit spreads to corporate-only QE is much stronger than to Treasury-only QE intervention (D'Amico & King, 2013; Krishnamurthy & Vissing-Jorgensen, 2011).

Related Literature This paper contributes to the literature on term structure models by integrating credit risk valuation into the preferred-habitat tradition. My model belongs to the general class of affine term structure models (Dai & Singleton, 2000; Duffie & Kan, 1996), and extends the preferred-habitat model of Vayanos and Vila (2021) by introducing defaultable bonds and deriving asset pricing implications for corporate bond yields and credit spreads. Building on Duffie and Singleton (2003) and Lando (1998), default events are idiosyncratic but default probabilities vary over time.

The preferred-habitat view of the term structure dates back to early work by Culbertson (1957) and Modigliani and Sutch (1966), but it was only recently formalized by Vayanos and Vila (2021).
My paper is closest to Costain, Nuño, and Thomas (2022) and Greenwood, Hanson, and Liao (2018), but there are important differences. As opposed to Costain et al. (2022), I study a portfolio of corporate bonds and assume that credit risk is diversifiable. This assumption is more suitable for the corporate bond market, and it has different implications for affine models of the term structure[8]. In particular, corporate and Treasury bonds are both affine functions of all the risk factors, including market specific risk factors such as the default intensity. The structure of financial markets is similar to Greenwood et al. (2018), with the key difference that monetary policy and fundamental shocks do affect the residual supply and that I derive implications for the entire term structure.
Greenwood and Vayanos (2014) explore supply effects on the term structure of Treasury yields, whereas Gourinchas et al. (2022) discuss a two country extension of Vayanos and Vila (2021) to explain the transmission of monetary policy across countries. Similarly, Greenwood, Hanson, Stein, and Sunderam (2020) develop a quantity theory of term premia and exchange rates, whereas Droste, Gorodnichenko, and Ray (2021) embeds habitat demand in a New Keynesian framework to explain the financial effects of QE. Beyond Treasury bonds, preferred-habitat models have been used in the repo (He, Nagel, & Song, 2022; Jappelli, Pelizzon, & Subrahmanyam, 2023), the MBS (Malkhozov, Mueller, Vedolin, & Venter, 2016), and the interest rate swaps markets (Hanson, Malkhozov, & Venter, 2022).

My paper also contributes to the literature on the transmission of monetary policy to long term rates. I argue that both duration and credit risk premia affect how monetary policy propagates to long term government and corporate rates. The sensitivity of demand to interest rates and default probabilities determine how risk prices vary in response to monetary policy and fundamental shocks. Vayanos and Vila (2021) only partially capture the evidence by Hanson and Stein (2015) that changes in long term forward rates overreact to changes in the policy rate on FOMC announcement days. Building

[8]See Jarrow, Lando, and Yu (2005) for a discussion on diversifiable vs. systematic default risk.

on this, Kekre, Lenel, and Mainardi (2024) integrate element from the intermediary asset pricing tradition into Vayanos and Vila (2021) to show that a monetary easing also revalues the wealth of the arbitrageurs. Closely related to this, Gertler and Karadi (2015) show that a monetary tightening is associated to an increase in various measures of credit spreads.

The theoretical analysis highlights another channel through which monetary policy affects risk premia on defaultable corporate debt. The channel works entirely through bond portfolios and does not rely on wealth effects or assumptions about capital structure policies and default decisions (H. Chen, 2010; Hackbarth, Miao, & Morellec, 2006). This mechanism complements other channels through which monetary policy moves bond risk premia, such as wealth redistribution (Auclert, 2019; Kekre & Lenel, 2022; Schneider, 2022), limited stock market participation (Alvarez, Atkeson, & Kehoe, 2009), and revaluation of intermediaries' wealth (Kekre et al., 2024).
Another important property of my model is that the endogenous variation in risk premia does not rely on restrictions on state dynamics. I also do not make any assumption on the dependence between agents' marginal utility and aggregate shocks, contrary to external habits (Wachter, 2006) and long-run risk models (Bansal & Shaliastovich, 2013). To emphasize that risk premia vary through movements in residual supply and portfolio composition, I shut down wealth effects by assuming CARA preferences. While Gârleanu and Panageas (2015) and Schneider (2022) study heterogeneity in risk aversion and intertemporal elasticity of substitution, I instead emphasize that demand for corporate and Treasury bonds is partially inelastic.

Lastly, my paper relates to the literature on the determinants of credit spreads changes and the credit spread puzzle. Collin-Dufresne et al. (2001) show that structural models of credit risk have limited explanatory power for changes in credit spreads. However, the unexplained part has a strong principal component. Friewald and Nagler (2019) and He, Khorrami, and Song (2022) link this common component to OTC frictions and intermediary capital, respectively. L. Chen et al. (2008) and Du et al. (2019) document that structural credit risk models underestimate credit spreads.
Finally, the model contributes to the literature on bond risk premia (e.g. Cochrane and Piazzesi (2005); Haddad and Sraer (2020)) by showing that when corporate and Treasury bonds are priced by the same marginal investors, asset specific risk factors may propagate to the other asset. This logic borrows from the intermediary asset pricing literature (Brunnermeier & Sannikov, 2014; He & Krishnamurthy, 2013). However, traditional factor models of corporate bonds generally treat interest rate and credit risk separately (Acharya, Amihud, & Bharath, 2013; Kelly, Palhares, & Pruitt, 2023).

Organization The remainder of the paper is organized as follows. Section 2 presents a tractable model of credit and duration risk premia in the preferred-habitat tradition. Section 3 calibrates the model in Section 2 to present a quantitative analysis of the key economic mechanisms. Section 4 explores the predictions of the theoretical analysis. Section 5 concludes.

2 Theoretical Framework

I propose an affine term structure model to study Treasury and corporate bond yields jointly. The core of the theory is a mechanism that connects interest rate risk, credit risk, and aggregate bond quantities. The key economic assumptions are that credit risk is diversifiable and that bonds are in non-zero net supply. I use the model to characterize the dynamics of credit spreads and bond risk premia and to study how monetary policy and fundamental shocks propagate across markets. All the proofs, additional lemmata, and model extensions are in Appendix A and Appendix C.

2.1 Environment

Timing and Assets Time t is continuous and runs from zero to infinity. Let $j \in \{g, c\}$ index the government and the corporate sector, respectively. A zero-coupon Treasury bond with maturity τ is a security that promises one unit of the numéraire at time $t+\tau$ with certainty, where $\tau \in (0, \infty)$ is time to maturity. The corporate sector is a continuum of identical firms issuing risky zero-coupon bonds with maturity $\tau \in (0, \infty)$. Within each maturity τ, there is a continuum of uniformly distributed bonds indexed by c_i, with $i \in [0, 1]$. For each bond i, a default event is an unpredictable jump in a Poisson process N_t^i with intensity λ_t^i. Given a default intensity λ_t^i, the probability of default within the interval $[t, t+dt]$ is $\lambda_t^i dt$. Although the default intensity does not depend on maturity, λ_t^i varies over time. The Poisson increment dN_t^i takes the value of one if bond i defaults and zero otherwise. Bond investors recover a constant fraction ω of market value upon default, which I set $\omega = 0$ for simplicity[1]. The corporate sector instantaneously issues new bonds to replace those that defaulted. I assume that defaults are idiosyncratic and that each bond has the same default intensity $\lambda_t^i = \lambda_t$ for all i.

Assumption 1 (Idiosyncratic Defaults)**.** *The increments dN_t^i are independent across i and all have the same time-varying default intensity λ_t.*

Contrary to Costain et al. (2022), credit events are diversifiable and not systematic. With idiosyncratic defaults, a deterministic fraction $\lambda_t dt$ of bonds defaults at any point in time. While there is no uncertainty about how many bonds default in the interval $[t, t+dt]$, the fraction of corporate bonds defaulting in the future is uncertain. Therefore, the credit risk premium is interpreted as a drift adjustment on the dynamics of default intensity. Assumption 1 also implies that the risk price associated to the default event is zero, and that default intensity λ_t is the same under both the empirical ($\mathbb{P}$) and the martingale ($\mathbb{Q}$) measure. This assumption reflects two principles[2]. First, most of the aggregate variation in credit spreads comes from discount rates rather than cash flow news (Nozawa, 2017). Second, the implicit equivalence between empirical and martingale default intensities simplifies the

[1]The results go through also with a constant fractional recovery of market value $\omega > 0$. The pricing expressions are, however, different when the recovery rate varies over time. See Duffie and Singleton (2003) for a discussion.

[2]Structural credit risk models typically do not generate strictly positive credit spreads at shorter horizons unless the assets of the firm follow jump-diffusion dynamics. An exception is Duffie and Lando (2001).

pricing of credit risk (Jarrow et al., 2005).

Let $P_{j,t}^{(\tau)}$ and $y_{j,t}^{(\tau)}$ be the price and the yield of asset class j with maturity τ at time t, respectively. Yields and prices are related through

$$y_{j,t}^{(\tau)} = -\frac{1}{\tau} \log P_{j,t}^{(\tau)}$$

and the instantaneous holding period return is $\frac{dP_{j,t}^{(\tau)}}{P_{j,t}^{(\tau)}}$. The instantaneous return on a well-diversified portfolio of defaultable bonds with maturity τ is defined to be

$$\frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}} \doteq \int_0^1 \frac{dP_{c_i,t}^{(\tau)}}{P_{c_i,t}^{(\tau)}} di$$

Finally, the short rate r_t is the limit of the yield $y_{g,t}^{(\tau)}$ as τ goes to zero and it is set exogenously by a monetary authority outside of the model.

Agents There are two types of agents: Arbitrageurs and preferred-habitat investors. Habitat investors, indexed by $\tau \in (0, \infty)$, are uniformly distributed across maturities and asset classes. I assume an extreme form of segmentation along both maturity and asset class akin to Greenwood et al. (2018), meaning that investors with habitat τ do not respond to prices of bonds outside their habitat. Although corporate bond markets are mostly segmented across credit ratings (Becker & Ivashina, 2015; Bessembinder, Jacobsen, Maxwell, & Venkataraman, 2018), maturity also plays a role, for example through maturity mandates (Bretscher, Schmid, & Ye, 2023). Within each maturity, both agents hold well-diversified portfolios of corporate bonds. Agents with habitat τ at time t hold

$$z_{g,t}^{(\tau)} = -\alpha^g(\tau) \log P_{g,t}^{(\tau)} + \gamma^g(\tau) \log P_{c,t}^{(\tau)} - \rho^g(\tau) r_t - \phi^g(\tau)\lambda_t - \beta_t^{g,(\tau)} \quad (1)$$
$$z_{c,t}^{(\tau)} = -\alpha^c(\tau) \log P_{c,t}^{(\tau)} + \gamma^c(\tau) \log P_{g,t}^{(\tau)} - \rho^c(\tau) r_t - \phi^c(\tau)\lambda_t - \beta_t^{c,(\tau)} \quad (2)$$

in bonds with maturity τ and hold no other position[3]. Habitat investors substitute across asset classes but only within the same maturity. The functions $\alpha^j(\tau) \geq 0$ and $\gamma^j(\tau)$, $j \in \{g, c\}$, which only depend on maturity, determine the own- and the cross-price elasticity of bond demand within habitat τ. If $\alpha^j(\tau) = \gamma^j(\tau)$, habitat demand is proportional to credit spreads, whereas $\alpha^j(\tau) \neq \gamma^j(\tau)$ captures imperfect substitutability between assets. I also allow habitat demand to respond directly to interest rates and default probabilities, with loadings $\rho^j(\tau)$ and $\phi^j(\tau)$, respectively. The first term introduces, in reduced-form, a component of demand that varies with the level of interest rates. This could happen, for example, through revaluation of existing bond positions (Kekre et al., 2024) or if some investors are yield oriented (Becker & Ivashina, 2015; Daniel et al., 2021). On the other hand, the second term captures habitat investors' appetite for credit risk. A deterioration in the fundamentals

[3]The link between credit and interest risk premia is not driven by habitat investors substituting across asset classes. As the calibration in Section 3 illustrates, the results go through also when $\gamma^j(\tau) = 0$. Central to the theory is that, in equilibrium, habitat demand is affine in the risk factors. A more general specification of the demand system that allows for substitution across habitats does not qualitatively affect the results, but complicates the analysis.

typically induces bonds sales from pension funds and insurance companies[4].
The intercept $\beta_t^{(\tau)}$ is time-varying and can depend on τ. I specify $\beta_t^{(\tau)}$ as

$$\beta_t^{j,(\tau)} = \theta_0^j(\tau) + \sum_{k=1}^{K} \theta_k^j(\tau)\beta_{k,t} \tag{3}$$

where the loadings $\{\theta_k^j(\tau)\}_{k=0}^K$ are constant over time but can depend on maturity τ. The specification of the demand intercept (3) is flexible, and it accommodates asset specific demand shocks. For example, suppose that the first and the second factors are pure Treasury and corporate bond demand shocks, respectively. Then, $\theta_{c,1}(\tau) = 0$ and $\theta_{g,2}(\tau) = 0$, so that $\beta_{1,t}$ ($\beta_{2,t}$) only affects habitat demand for Treasury (corporate) bonds. These demand factors can also be interpreted as shocks to the residual supply as in Greenwood and Vayanos (2014) and He, Nagel, and Song (2022).

Arbitrageurs trade corporate bonds and Treasury bonds at all maturities and can invest in a risk-free asset that pays the short rate r_t. Let W_t and $x_{j,t}^{(\tau)}$ denote arbitrageurs' wealth and dollar holdings in bond $j \in \{g, c\}$ with maturity τ. Arbitrageurs have mean-variance preferences over instantaneous changes in wealth

$$\max_{\{x_{j,t}^{(\tau)}\}_{\tau\in\{0,\infty\}}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)\right] \tag{4}$$

where $a \geq 0$ is the arbitrageurs' coefficient of absolute risk aversion. The budget constraint is

$$dW_t = \left(W_t - \int_0^\infty \sum_j x_{j,t}^{(\tau)} d\tau\right) r_t dt + \int_0^\infty x_{g,t}^{(\tau)} \frac{dP_{g,t}^{(\tau)}}{P_{g,t}^{(\tau)}} d\tau + \int_0^\infty x_{c,t}^{(\tau)} \frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}} d\tau \tag{5}$$

The first term in equation (5) corresponds to a position in the risk-free asset, the second term to a position in Treasury bonds, and the third term to a position in a well-diversified portfolio of defaultable bonds. The case of segmented arbitrage along the lines of Gourinchas et al. (2022) can be obtained by assuming that arbitrageurs can only trade in one market (e.g. Treasury). The instantaneous return on the bond index is $\frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}}$, which implicitly accounts for the defaults in the interval $[t, t + dt]$.

Risk Factor Dynamics There are $K + 2$ risk factors. The aggregate risk factors are the short rate r_t, the default intensity λ_t, and the K demand factors $\beta_{k,t}$ for $k = 1, \ldots K$. The $(K + 2) \times 1$ vector $s_t \doteq (r_t, \lambda_t, \beta_{1,t}, \ldots, \beta_{K,t})^T$ follows the homoscedastic mean-reverting process

$$ds_t = -\Gamma(s_t - \bar{s})dt + \Sigma dB_t \tag{6}$$

where $\bar{s}$ is a $(K + 2) \times 1$ vector of long-term averages and $dB_t = (dB_{r,t}, dB_{\lambda,t}, dB_{\beta,1,t}, \ldots, dB_{\beta,K,t})^T$ is a $(K + 2) \times 1$ vector of independent Brownian motions. The matrix Γ controls the speed of mean reversion, whereas the instantaneous covariance matrix is $\Sigma\Sigma^T$. Equation (6) nests the special case in which Γ and Σ are diagonal and the risk factors are independent.

[4]In Appendix A.1, I discuss an optimizing microfoundation of habitat demand with stochastic volatility.

The dynamics in equation (6) are homoscedastic and the parameters Σ and Γ are constant over time. My goal is to study variation in the sign and magnitude of risk prices for a given specification of the state dynamics, and understand how shocks propagate across markets. Yet, the choices of Γ and Σ have direct implications for how risk premia move with the state variables, regardless of whether bonds are in non-zero net supply. Indeed, a common approach to generate variation in risk premia over time is to consider stochastic volatility (Bansal & Shaliastovich, 2013) or time-varying risk prices through, for example, external habits (Wachter, 2006). Many of these approaches require *ex-ante* restrictions on the covariance between the pricing kernel and the risk factors to generate risk premia with the desired properties. Further, variation in risk premia is typically proportional to the quantity of risk. This makes it difficult to characterize how and why risk prices vary with bond quantities, or how they depend on asset substitutability.

Market Clearing Bond markets for $j \in \{g, c\}$ clear at each maturity τ

$$z_{j,t}^{(\tau)} + x_{j,t}^{(\tau)} = 0$$

at each point in time. The equilibrium is a collection of prices and quantities $\left\{P_{j,t}^{(\tau)}, x_{j,t}^{(\tau)}\right\}_{\tau \in (0,\infty)}$ such that arbitrageurs' are optimizing and markets clear for all maturities τ and assets j.

2.1.1 Equilibrium with Arbitrageurs

I conjecture that yields of both government and corporate bonds are affine functions of the state variables. In particular, there exists functions $(A_j(\tau)^T, C_j(\tau))$ for $j \in \{g, \{c_i\}_{i\in[0,1]}\}$ such that

$$P_{j,t}^{(\tau)} = e^{-\left[A_j(\tau)^T s_t + C_j(\tau)\right]} \tag{7}$$

Importantly, conjecture (7) states that Treasury yields also load on default intensity λ_t, irrespective of the state dynamics. Under conjecture (7), the instantaneous return on Treasury bonds is

$$\begin{aligned}
\frac{dP_{g,t}^{(\tau)}}{P_{g,t}^{(\tau)}} &= \mu_{g,t}^{(\tau)} dt - A_g(\tau)^T \Sigma dB_t \\
\mu_{g,t}^{(\tau)} &= A_g'(\tau)^T s_t + C_g'(\tau) + A_g(\tau)^T \Gamma (s_t - \overline{s}) + \frac{1}{2} A_g(\tau)^T \Sigma \Sigma^T A_g(\tau)
\end{aligned}$$

The instantaneous return on each individual defaultable bond i is

$$\frac{dP_{c_i,t}^{(\tau)}}{P_{c_i,t}^{(\tau)}} = \left[1 - dN_t^i\right] \left(\mu_{c_i,t}^{(\tau)} dt - A_{c_i}(\tau)^T \Sigma dB_t\right) + dN_t^i(\omega - 1) = \mu_{c_i,t}^{(\tau)} dt - A_{c_i}(\tau)^T \Sigma dB_t - dN_t^i$$

where

$$\mu_{c_i,t}^{(\tau)} = A_{c_i}'(\tau)^T s_t + C_{c_i}'(\tau) + A_{c_i}(\tau)^T \Gamma (s_t - \overline{s}) + \frac{1}{2} A_{c_i}(\tau)^T \Sigma \Sigma^T A_{c_i}(\tau)$$

and I set $\omega = 0$. The last equality holds because the cross-variation between a Brownian motion and a point process is zero. Since there is no counterparty risk and defaults are idiosyncratic, the bonds are ex-ante identical. Hence, in equilibrium it must be that $\mu^{(\tau)}_{c_i,t} = \mu^{(\tau)}_{c,t}$ and $\sigma^{(\tau)}_{c_i,t} = \sigma^{(\tau)}_{c,t}$. Hence

$$\frac{dP^{(\tau)}_{c,t}}{P^{(\tau)}_{c,t}} \doteq \int_0^1 \frac{dP^{(\tau)}_{c_i,t}}{P^{(\tau)}_{c_i,t}} di = \mu^{(\tau)}_{c,t} dt + \sigma^{(\tau)}_{c,t} dB_t - \lambda_t dt \tag{8}$$

where the second equality follows from the Law of Large Numbers, as shown in Appendix C.3. Plugging instantaneous returns into the arbitrageurs' budget constraint gives the objective

$$\max_{\{x^{(\tau)}_{j,t}\}_{\tau \in (0,\infty)}} \left(W_t - \int_0^\infty \sum_j x^{(\tau)}_{j,t} \right) r_t dt + \int_0^\infty \mu^{(\tau)}_{j,t} x^{(\tau)}_{g,t} d\tau dt + \int_0^\infty \left(\mu^{(\tau)}_{c,t} - \lambda_t \right) x^{(\tau)}_{c,t} d\tau dt$$
$$- \frac{a}{2} \left[\int_0^\infty \sum_j x^{(\tau)}_{j,t} A_j(\tau)^T d\tau \right] \Sigma\Sigma^T \left[\int_0^\infty \sum_j x^{(\tau)}_{j,t} A_j(\tau) d\tau \right] dt$$

Equation (8) implies that there are no jumps in the return of the bond portfolio held by arbitrageurs, so that the event risk premium is zero. Because event risk is fully diversifiable, a well-diversified bond portfolio with maturity τ will have the same price as each of its components prior to default. Pointwise maximization with respect to $x^{(\tau)}_{j,t}$ yields the set of first-order conditions

$$\mu^{(\tau)}_{g,t} - r_t = A_g(\tau)^T \Sigma \cdot \eta_t \tag{9}$$
$$\mu^{(\tau)}_{c,t} - r_t = \lambda_t + A_c(\tau)^T \Sigma \cdot \eta_t \tag{10}$$

where

$$\eta_t = a\Sigma^T \left[\sum_j \int_0^\infty x^{(\tau)}_{j,t} A_j(\tau) d\tau \right] \tag{11}$$

is the vector of risk prices. Absence of arbitrage does not say anything about what the risk prices should be. These prices are instead determined through market clearing. The residual supply is affine in the state variables, and so is the vector of risk prices η_t.

The first-order conditions (9) and (10) pin down bond excess returns, and reflect absence of arbitrage in continuous time. As in Vayanos and Vila (2021), there exist prices specific to each risk factor and common across assets, such that the expected excess return on any asset is equal to the sum across factors of the asset sensitivity to each factor times the factor's price. Bond risk premia depend on the arbitrageurs' aggregate bond positions in the Treasury and the corporate market. Shocks in the corporate bond market propagate to Treasury yields, affecting their excess returns, and vice versa. The main difference between (9) and (10) is that the default intensity λ_t only shows up directly in the first-order condition for corporate bonds, reflecting the expected default component of credit spreads (Gilchrist & Zakrajšek, 2012). The coefficient in front of λ_t is one, meaning that the market price associated to the default event is zero. The quantity $\left[\sum_j \int_0^\infty x^{(\tau)}_{j,t} A_j(\tau) d\tau\right]$ can be interpreted as arbitrageurs' inventories in the spirit of He, Khorrami, and Song (2022). Although expected returns load differently on this common factor through asset specific sensitivities $A_j(\tau)$, the first-order conditions suggest that a strong principal component is likely to capture most of the variation in bond expected returns over and above variation in expected defaults. This observation is consistent with Friewald and Nagler (2019) and He, Khorrami, and Song (2022).

Under the exponential-affine conjecture, habitat demand is

$$z_{j,t}^{(\tau)} = \left\{\alpha^j(\tau)C_j(\tau) - \gamma^j(\tau)C_{-j}(\tau) - \theta_0^j(\tau)\right\} + \left[\alpha^j(\tau)A_j(\tau)^T - \gamma^j(\tau)A_{-j}(\tau)^T - \Theta^j(\tau)\right] s_t$$

where the $1 \times (K+2)$ vector $\Theta^j(\tau)$ is defined as $\Theta^j(\tau) \doteq (\rho^j(\tau), \phi^j(\tau), \theta_1^j(\tau), \ldots, \theta_K^j(\tau))$. Plugging the market clearing conditions $x_{j,t}^{(\tau)} = -z_{j,t}^{(\tau)}$ back into the arbitrageurs' first-order conditions and matching coefficients on s_t delivers two systems of $K+2$ linear first-order ordinary differential equations

$$A_g'(\tau) + MA_g(\tau) - e_1 = 0 \tag{12}$$

$$A_c'(\tau) + MA_c(\tau) - e_1 - e_2 = 0 \tag{13}$$

where the $(K+2) \times (K+2)$ matrix M is

$$M \doteq \Gamma^T - a \sum_j \int_0^\infty \left[\Theta^j(\tau)^T - \alpha^j(\tau)A_j(\tau) + \gamma^j(\tau)A_{-j}(\tau)\right] A_j(\tau)^T d\tau \Sigma\Sigma^T \tag{14}$$

The matrix M captures state-dependent risk adjustments, and it is the counterpart of Γ under the pricing measure. If arbitrageurs' are risk-neutral ($a = 0$), then $M^T = \Gamma$. When $a = 0$, arbitrageurs do not require any compensation for risk, and excess bond returns are zero for all assets, that is $\mu_{g,t}^{(\tau)} - r_t = 0$ and $\mu_{c,t}^{(\tau)} - r_t = \lambda_t$. The summation term shows that bond risk premia depend on the aggregate holdings of Treasury and corporate bonds and their exposure to s_t. This is because, in equilibrium, both asset classes are exposed to duration and credit risk.
I solve (12) and (13) with the boundary conditions $A_g(0) = A_c(0) = 0$. The next Proposition characterizes the equilibrium with K demand shocks and general state dynamics.

Proposition 1 (Equilibrium). *Given boundary conditions $A_g(0) = A_c(0) = \mathbf{0}$, the $(K+2)$ functions $A_g(\tau) = (A_{g,r}(\tau), A_{g,\lambda}(\tau), \{A_{g,\beta_k}(\tau)\}_{k=1}^K)^T$ and $A_c(\tau) = (A_{c,r}(\tau), A_{c,\lambda}(\tau), \ldots, \{A_{c,\beta_k}(\tau)\}_{k=1}^K)^T$ are given by*

$$A_g(\tau) = \sum_{k=1}^{K+2} \psi_k^g \left(\frac{1 - e^{-\mathsf{v}_k \tau}}{\mathsf{v}_k}\right) \tag{15a}$$

$$A_c(\tau) = \sum_{k=1}^{K+2} \psi_k^c \left(\frac{1 - e^{-\mathsf{v}_k \tau}}{\mathsf{v}_k}\right) \tag{15b}$$

where v_k are the eigenvalues of M defined in (14). Furthermore, ψ_k^j are vectors such that $\psi_k^j = \mathsf{u}_k \xi_i^j$, where u_k is the eigenvector corresponding to v_k and ξ_k^j is the asset-specific k-th component of $\xi^j \doteq P^{-1}\boldsymbol{b}^j$, where $P \doteq [\mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{K+2}]$, $\boldsymbol{b}^g = e_1$ and $\boldsymbol{b}^c = e_1 + e_2$.

What makes the system hard to characterize is that the elements of M involve integrals of $A_j(\tau)$, which depend on the eigenvectors and eigenvalues of M itself. Furthermore, the matrix M has to

make sure that both (15a) and (15b) hold simultaneously.
Taking the solution of $A_g(\tau)$ and $A_c(\tau)$ as given, I collect constant terms and obtain

$$C_g(\tau) = \left[\int_0^\tau A_g(u)^T du\right]\chi - \frac{1}{2}\int_0^\tau A_g^T(u)\Sigma\Sigma^T A_g(u)du \tag{16a}$$

$$C_c(\tau) = \left[\int_0^\tau A_c(u)^T du\right]\chi - \frac{1}{2}\int_0^\tau A_c^T(u)\Sigma\Sigma^T A_c(u)du \tag{16b}$$

where χ is a $K+2$ vector of constants such that

$$\chi \doteq \Gamma\bar{s} + a\Sigma\Sigma^T\left(\sum_j \int_0^\infty \left[\theta_0^j(\tau) - \alpha^j(\tau)C_j(\tau) + \gamma^j(\tau)C_{-j}(\tau)\right]A_j(\tau)d\tau\right) \tag{17}$$

To solve for the vector of constants χ, I substitute (16a) and (16b) into (17) and derive a system of $K+2$ equations in the $K+2$ unknown entries of χ. The vector χ captures the component of risk prices that is constant over time. This component is non-zero even when residual supply is price inelastic.

Economic Interpretation of Risk Prices Proposition (1) shows that Treasury and corporate bond yields are affine in the risk factors. An interpretation of the pricing formulas in Proposition (1) in terms of risk-neutral pricing is immediate[5], since

$$P_{g,t}^{(\tau)} = e^{-[A_{gr}(\tau)r_t + A_{g\lambda}(\tau)\lambda_t + C_g(\tau)]} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} r_u du}\right]$$

$$P_{c,t}^{(\tau)} = e^{-[A_{cr}(\tau)r_t + A_{c\lambda}(\tau)\lambda_t + C_c(\tau)]} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau}(r_u+\lambda_u)du}\right]$$

and the martingale dynamics of s_t are given by

$$ds_t = -M^T\left(s_t - \bar{s}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}} \tag{18}$$

where $\bar{s}^{\mathbb{Q}}$ is implicitly defined by $M^T\bar{s}^{\mathbb{Q}} = \chi$.

It turns out that risk prices are affine in s_t, that is $\eta_t = \eta_0 + \eta_1 s_t$. Both the speed of mean-reversion Γ and the long term average $\bar{s}$ are different under the pricing measure. On the one hand, the vector χ is related to η_0. In contrast, the matrix M captures how risk prices vary with the state variables, i.e. η_1. Typically, η_1 is non-diagonal, even in the special case that the risk factors are independent and regardless of any restriction on the covariance structure of the shocks. In summary, movements in the Treasury term premia can generate from movements in both interest rate and credit risk prices.
As opposed to reduced-form models of credit risk valuation, I first specify state dynamics under the empirical measure. The standard approach in valuing credit risk is to directly model the martingale dynamics of the risk factors, taking a stance on risk prices (see e.g. Duffie and Singleton (2003, 1999)). Many models are flexible enough to accommodate risk-neutral dependence between r_t and λ_t, but assumptions on the martingale dynamics usually build on historical correlations. As such, they impose *ex-ante* restrictions on the dynamics of risk premia. The dependence between r_t and λ_t

[5] I derive the connection to standard affine term structure models explicitly in Appendix C.2.2.

is often exogenously specified, explicitly through modelling of risk prices (Dai & Singleton, 2002) or implicitly through common loadings on a latent business cycle factors (Duffie & Singleton, 2003). For example, defaults are more likely to occur in bad times, precisely when interest rates are lower. In my model, this dependence emerges endogenously through the bond portfolios of arbitrageurs.

Equation (18) highlights a novel channel that naturally links interest rate risk and credit risk prices when corporate and Treasury bonds are in non-zero net supply. The price of interest rate risk varies with the short rate r_t and the default intensity λ_t. The same holds for the price of credit risk. As a result, arbitrageurs' portfolios endogenously connect the duration and the credit risk premia. As shown in equation (11), the aggregate residual supply of both Treasury and corporate bonds determines duration and credit risk prices. Intuitively, this is because both Treasuries and corporate bonds carry exposure to duration risk. Demand shocks that originate in the corporate bond market propagate to Treasury yields through this common exposure, and vice versa.
Equation (14) and (17) show that bond demand and asset substitutability are an important determinant of the equilibrium relation between the price of interest rate and credit risk and the risk factors. This occurs because movements in the composition of the arbitrageurs' portfolio are driven by how sensitive habitat demand is to changes in interest rates and default probabilities. Changes in asset demand and bond substitutability impact the quantity of risk in Treasury and corporate bonds. For example, an increase in $\phi^c(\tau)$ makes corporate bond riskier because arbitrageurs have to accommodate selling pressure from habitat investors precisely when credit quality deteriorates. Another example is the introduction of capital requirements that makes Treasury demand less elastic.

2.2 Propagation of Shocks Across Markets

I study how monetary policy and fundamental shocks propagate across markets. To simplify the analysis and emphasize the key economic mechanism, I specialize Γ and Σ to be diagonal. I abstract from demand shocks, so that the short rate and the default intensity are the only risk factors.

2.2.1 Treasury Yields and Credit Risk

I first establish that the loading of Treasury yields on default intensity $A_{g\lambda}(\tau)$ is non-zero for all τ.

Proposition 2 (Treasury Loading on Default Intensity)**.** *Suppose that $K = 0$ and that Σ and Γ are diagonal. Then the loadings of Treasury yields on default intensity λ_t is*

$$A_{g\lambda}(\tau) = \frac{\kappa_{r\lambda}}{\nu_2 - \nu_1}\left(\frac{1 - e^{-\nu_1 \tau}}{\nu_1} - \frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right)$$

where $\kappa_{r\lambda} = -M_{21}$ describes how the price of duration risk varies with λ_t and it is given by

$$\kappa_{r\lambda} = a\sigma_r^2 \int_0^\infty \sum_j \left(\phi^j(\tau) - \alpha^j(\tau)A_{j\lambda}(\tau) + \gamma^j(\tau)A_{-j\lambda}(\tau)\right) A_{jr}(\tau)d\tau$$

Further, the eigenvalues v_1 *and* v_2, $\mathsf{v}_1 > \mathsf{v}_2$ *are*

$$\mathsf{v}_{1,2} = \frac{\kappa_r^* + \kappa_\lambda^* \pm \sqrt{(\kappa_r^* - \kappa_\lambda^*)^2 - 4\kappa_{\lambda r}\kappa_{r\lambda}}}{2}$$

where κ_r^*, κ_λ^*, *and* $\kappa_{\lambda r}$ *are given in Appendix C.*

Since the eigenvectors of M are usually distinct, i.e. $\mathsf{v}_1 > \mathsf{v}_2$, the loading $A_{g\lambda}(\tau)$ is zero for all τ only if $\kappa_{r\lambda} = -M_{21}$ is zero. Proposition (2) shows that Treasury yields load on default intensity even when r_t and λ_t are independent. This occurs because the price of interest rate risk varies with λ_t. The direction of the effect depends on whether a deterioration in credit quality, that is an increase in λ_t, raises or lowers the price of interest rate risk. In the special case that interest rates are constant ($\sigma_r = 0$) or when arbitrageurs are risk neutral ($a = 0$), then $A_{g\lambda}(\tau) = 0$. The term $\kappa_{r\lambda}$ is also zero when residual supply is independent of λ_t, although this extreme case requires $\phi^j(\tau) = \alpha^j(\tau) = \gamma^j(\tau) = 0$ for all τ and for $j \in \{g, c\}$.

Proposition (2) describes the key economic mechanism through which shocks to default rates propagate to the Treasury market. To build intuition, I consider a special case in which habitat investors are price inelastic and substitute Treasury bonds for corporate bonds one-to-one in response to changes in λ_t[6]. Then, $\kappa_{r\lambda}$ is given by

$$\kappa_{r\lambda} = a\sigma_r^2\phi\left[\left(\int_0^\infty A_{cr}(\tau)d\tau\right) - \left(\int_0^\infty A_{gr}(\tau)d\tau\right)\right]$$

Since $a\sigma_r^2\phi > 0$, the sign of $\kappa_{r\lambda}$ entirely depends on whether the aggregate exposure of corporate bonds to interest rates is larger or smaller than the aggregate exposure of Treasuries to interest rates. Typically, Treasury bonds have longer duration, so that $A_{gr}(\tau) > A_{cr}(\tau) > 0$ at all τ and $\kappa_{r\lambda} < 0$. Since $\mathsf{v}_1 > \mathsf{v}_2$ it follows that, at least for small τ, $A_{g\lambda}(\tau) < 0$. In this example, a deterioration in credit quality lowers the price of interest rate risk because the increase in habitat demand for Treasury bonds reduces the aggregate quantity of duration arbitrageurs hold in equilibrium. Increases in arbitrageurs' risk aversion or in the volatility of interest rate shocks amplify this effect.

2.2.2 Monetary Policy Transmission to Corporate Yields

The interaction of credit and interest rate risk has important implications for the transmission of monetary policy to long term corporate rates. This is important because investment and credit supply decisions are made by firms and financial intermediaries, which cannot usually borrow at the same rate as the government.

Building on Proposition (2), I next argue that the level of default uncertainty σ_λ has implications for the transmission of monetary policy to long rates. The effect of an increase in σ_λ is potentially asymmetric across markets, and depends on whether $A_{g\lambda}(\tau)$ and $A_{c\lambda}(\tau)$ have the same or the opposite sign. Since σ_λ is constant and changes in volatility are outside of the model, however, these comparisons should be interpreted as comparative statics.

[6]Formally, habitat demand is $z_{j,t}^{(\tau)} = -\rho^j(\tau)r_t - \phi^j(\tau)\lambda_t - \theta_0^j(\tau)$, and $\phi^g(\tau) = -\phi^c(\tau) = \phi > 0$ is constant.

As in Vayanos and Vila (2021), I assess monetary policy transmission to long rates by comparing the reaction of forward rates to that of the expected future short rates $\mathbb{E}_t[r_{t+\tau}]$. Instantaneous forward rates are defined as

$$f_{j,t}^{(\tau)} = \lim_{\Delta\tau \to 0} f_{j,t}^{(\tau-\Delta\tau,t)} = -\frac{\partial \log P_{j,t}^{(\tau)}}{\partial \tau} = A'_{jr}(\tau) r_t + A'_{j\lambda}(\tau)\lambda_t + C'_j(\tau)$$

Under the expectations hypothesis (EH), forward rates move one-to-one with expected future short rates. Conversely, if risk prices vary with either r_t or λ_t, the expectations hypothesis fails and transmission to long term bond yields is either partial or amplified.

Proposition 3 (Forward Rates Responses)**.** *Suppose that $K = 0$ and that Σ and Γ are diagonal. A unit shock to the short rate r_t raises the expected short rate $\mathbb{E}_t[r_{t+\tau}]$ by $e^{-\kappa_r \tau}$. In addition, the response of Treasury instantaneous forward rates for maturity τ is*

$$\frac{\partial f_{g,t}^{(\tau)}}{\partial r_t} = A'_{gr}(\tau) = \frac{\kappa_\lambda^* - \mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1} e^{-\mathsf{v}_1 \tau} - \frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1} e^{-\mathsf{v}_2 \tau}$$

where

$$\kappa_\lambda^* = \kappa_\lambda - a\sigma_\lambda^2 \int_0^\infty \sum_j \left(\phi^j(\tau) - \alpha^j(\tau) A_{j\lambda}(\tau) + \gamma^j(\tau) A_{-j\lambda}(\tau) \right) A_{j\lambda}(\tau) d\tau$$

and $\mathsf{v}_1, \mathsf{v}_2$ are as in Proposition (2). The response of corporate instantaneous forward rates is

$$\frac{\partial f_{c,t}^{(\tau)}}{\partial r_t} = A'_{cr}(\tau) = A'_{gr}(\tau) - \frac{\kappa_{\lambda r}}{\mathsf{v}_2 - \mathsf{v}_1} \frac{\kappa_\lambda^* - \mathsf{v}_2}{\kappa_r^* - \mathsf{v}_1} \left(e^{-\mathsf{v}_1 \tau} - e^{-\mathsf{v}_2 \tau} \right) \tag{19}$$

Proposition (3) shows that monetary policy does not affect long term Treasury and corporate bond yields in the same way. In particular, I find that

$$\frac{\partial f_{g,t}^{(\tau)}}{\partial r_t} \neq \frac{\partial f_{c,t}^{(\tau)}}{\partial r_t}$$

which means that the response of corporate and Treasury bond risk premia to a unit increase in r_t is heterogeneous. In a similar logic to Proposition (2), the wedge is proportional to $\kappa_{\lambda r} = -M_{12}$ and varies with arbitrageurs' risk aversion. Equation (19) implies that corporate yields overreact (underreact) to monetary policy shocks vis-à-vis Treasury yields when an increase in the short rate raises (lowers) the price of credit risk, that is when $\kappa_{\lambda r} < 0$. The magnitude of the wedge is proportional to default uncertainty σ_λ. It follows that comparing the responses of Treasury and corporate forward rates is informative on whether monetary policy impacts the price of credit risk.

More subtly, the volatility of default intensity shocks also affects transmission to Treasury yields vis-à-vis changes in the future expected short rate. When $A_{g\lambda}(\tau) > 0$, Treasury yields are positively related to default intensity, and an increase in σ_λ makes arbitrageurs' trades even riskier. Because of this, monetary policy transmission to long rates is weaker when default uncertainty is high. By contrast, if $A_{g\lambda}(\tau) < 0$, then Treasuries offer protection against default risk. Because hedging properties are more valuable when volatility σ_λ is high, an increase in default uncertainty lowers Treasury excess returns.

2.3 Credit Spreads and Shock Amplification

Proposition (1) delivers an expression that informs how monetary policy and demand shocks affect credit spreads. In equilibrium, yields for $j \in \{g, c\}$ are given by

$$y_{j,t}^{(\tau)} = \frac{1}{\tau}\left[A_j(\tau)^T s_t + C_j(\tau)\right]$$

The credit spread $\mathcal{S}_t^{(\tau)}$ at maturity τ is defined as the yield on corporate bonds minus the yield on Treasury bonds of the same maturity, that is

$$\mathcal{S}_t^{(\tau)} \doteq y_{c,t}^{(\tau)} - y_{g,t}^{(\tau)} = \frac{1}{\tau}\left[A_{\mathcal{S}}(\tau)^T s_t + C_{\mathcal{S}}(\tau)\right] \tag{20}$$

where $A_{\mathcal{S}} \doteq A_c(\tau) - A_g(\tau)$ and $C_{\mathcal{S}} \doteq C_c(\tau) - C_g(\tau)$. Let $\delta_t \doteq (\beta_{1,t}, \ldots, \beta_{k,t})$ be the vector of demand shocks. Credit spreads can then be written as

$$\mathcal{S}_t^{(\tau)} = \frac{1}{\tau}\left[A_{\mathcal{S}r}(\tau) r_t + A_{\mathcal{S}\lambda}(\tau)\lambda_t + A_{\mathcal{S}\delta}(\tau)\delta_t + C_{\mathcal{S}}(\tau)\right] \tag{21}$$

where $A_{\mathcal{S}\delta}(\tau) \doteq (A_{c\beta_1}, \ldots, A_{c\beta_k})^T - (A_{g\beta_1}, \ldots, A_{g\beta_k})^T$. Equation (21) reveals that credit spreads are affine functions of the risk factors in s_t. An immediate consequence is that credit spreads not only depend on expected default rates, but also on the level of the short rate r_t and all the other asset specific demand shocks. To the extent that $A_{\mathcal{S}r}(\tau) \neq 0$, $A_{\mathcal{S}\lambda}(\tau) \neq 0$, and $A_{\mathcal{S}\delta}(\tau) \neq 0$, changes in credit spreads are driven by either fluctuations in the credit quality of the corporate sector λ_t, movements of the short term rate r_t, and local or global demand effects δ_t.

I once again specialize the model to $K = 0$ and independent risk factors to describe how credit spreads respond to monetary policy and fundamental shocks.

Proposition 4 (Amplification of Credit Shocks)**.** *Suppose that $K = 0$ and that Σ and Γ are diagonal. Credit spreads $\mathcal{S}_t^{(\tau)}$ satisfy*

$$\tau\mathcal{S}_t^{(\tau)} = \left[A_{cr}(\tau) - A_{gr}(\tau)\right] r_t + \left[A_{c\lambda}(\tau) - A_{g\lambda}(\tau)\right]\lambda_t + C_c(\tau) - C_g(\tau)$$

where

$$A_{c\lambda}(\tau) - A_{g\lambda}(\tau) = -\frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1}\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\kappa_\lambda^* - \mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1}\frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}$$

In the special case that $a = 0$, then credit spreads are $\tau\mathcal{S}_t^{(\tau)} = \frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda}\lambda_t$. If $a = 0$, credit spreads do not depend on the short term rate r_t.

Proposition (4) states that changes in risk premia amplify (or mitigate) the effects of an increase in default probabilities on credit spreads. If $A_{c\lambda}(\tau) - A_{g\lambda}(\tau) > 1$, then a unit increase in λ_t moves $\tau\mathcal{S}_t^{(\tau)}$ more than one-to-one. This occurs mostly because the price of credit risk increases with λ_t. A secondary effect is that changes in λ_t also impact the market price of interest rate risk. If $|A_{c\lambda}(\tau) - A_{g\lambda}(\tau)| < 1$, an increase in default intensity is only partially incorporated into credit spreads. This occurs when the price of credit risk is inversely related to λ_t.

Consider now a linear regression of changes in credit spreads onto changes in the short rate and changes in default rates, keeping the maturity constant, that is

$$\Delta \mathcal{S}_t^{(\tau)} = \beta_0^{(\tau)} + \beta_1^{(\tau)} \Delta r_t + \beta_2^{(\tau)} \Delta \lambda_t + \varepsilon_t^{(\tau)}$$

Since the model implies that $\tau \beta_2^{(\tau)} = A_{c\lambda}(\tau) - A_{g\lambda}(\tau)$, coefficient estimates of $\beta_2^{(\tau)}$ are informative about the relation between the price of credit risk and default probabilities.

2.4 Supply Effects and Quantitative Easing

Credit and interest rate risk prices depend on the aggregate supply of Treasury and corporate bonds. The next result describes how shocks to the residual supply of one asset affects expected excess returns on the other asset as well as credit spreads. In the model, an increase in the residual supply of asset j is an exogenous shift in the intercept of habitat demand $\theta_0^j(\tau)$ for all τ. Changes in the intercept impact bond risk premia through the time-invariant component of risk prices η_0. Intuitively, supply shocks in the Treasury (corporate) market also affect corporate (Treasury) bonds expected excess returns, and the strength of the effect depends on each asset aggregate exposure to duration and credit risk.

Proposition 5 (Supply Effects)**.** *Suppose that $K = 0$ and that Σ and Γ are diagonal. Then, the difference between corporate and bond expected excess returns is*

$$\mu_{c,t}^{(\tau)} - \mu_{g,t}^{(\tau)} = \lambda_t + \sigma_r [A_{cr}(\tau) - A_{gr}(\tau)]\eta_{r,t} + \sigma_\lambda [A_{c\lambda}(\tau) - A_{g\lambda}(\tau)]\eta_{\lambda,t}$$

where $\eta_t = \eta_0 + \eta_1 s_t$, and η_{0s}, $s \in \{r, \lambda\}$ is

$$\eta_{0s} = a\sigma_s \sum_j \int_0^\infty \theta_0^j(\tau) A_{js}(\tau) d\tau + a\sigma_s \sum_j \int_0^\infty \left[\gamma^j(\tau) C_{-j}(\tau) - \alpha^j(\tau) C_j(\tau)\right] A_{js}(\tau) d\tau$$

Proposition (5) shows that changes in the demand intercept $\theta_0^j(\tau)$ affect risk prices of both credit and interest rate risk through the time-invariant component η_0. The direction of the effect of a proportional and permanent change of $\theta_0^j(\tau)$ to $\Delta\theta_0^j(\tau)$ for all τ depends on the aggregate exposure of asset j to risk factor s, that is $a\sigma_s \int_0^\infty A_{js}(\tau) d\tau$. If $\int_0^\infty A_{js}(\tau) d\tau > 0$, a reduction in net supply lowers the risk premium on factor s. However, if $\int_0^\infty A_{js}(\tau) d\tau < 0$, a reduction in supply can permanently raise the risk price $\eta_{t,s}$. Proposition (5) shows that a decline in Treasury supply, for example through quantitative easing (QE) programs, can raise credit spreads if the reduction in the duration risk premium disproportionately impacts Treasury yields. In contrast, a decline in the net supply of risky debt reduces both the interest rate and the credit risk premium, lowering credit spreads.

When Treasury bonds hedge against default risk, that is $A_{\lambda g}(\tau) < 0$ for all τ, QE interventions can adversely impact corporate yields by raising the market price of credit risk. Intuitively, Treasury holdings protect arbitrageurs against deteriorating economic fundamentals. A decline in Treasury supply makes safe bonds scarcer and reduces the availability of hedges against fundamental shocks, increasing arbitrageurs' average exposure to credit risk. This result is reminiscent of Krishnamurthy and Vissing-Jorgensen (2012), but the underlying mechanism is different. Bond quantities affect credit spreads through their effect on the credit risk premium, not necessarily by impacting the equilibrium price of convenience[7]. Overall, the interaction of credit and interest rate risk prices has implications for the transmission of non-conventional monetary policy. The effects of QE on corporate yields and credit spreads depend on the bundle of assets being purchased.

2.5 Discussion of Modelling Assumptions

Lastly, I discuss the key assumptions of the model and explore immediate extensions.

2.5.1 Substitution Patterns

I assume an extreme form of preferences for specific maturities. However, it seems reasonable that habitat investors should either (i) be responsive to the prices of bonds with very close maturities $\tau \pm d\tau$ or (ii) not be responsive at all. Vayanos and Vila (2021) provide an optimizing microfoundation based on infinitely large risk aversion and max-min preferences. Nevertheless, it seems difficult to connect this nonstandard behavior to institutional investors such as insurance companies and pension funds. A more realistic microfoundation should take into account specific mandates or constraints faced by these investors, incorporating duration matching, benchmarking, or regulations, for example. To the extent that a better microfounded demand function makes habitat investors respond to price of other bonds, habitat investors would partially behave as arbitrageurs themselves, increasing arbitrage capacity in the economy. The key propositions still hold in a more general framework, but the analysis must then consider a continuum of portfolios at the expense of tractability.

2.5.2 Homoscedastic Demand Shocks

The dynamics in equation (6) have the drawback that default intensity might become negative with non-zero probability. In simulations the probability that $\lambda_t < 0$ is negligible, but it seems sensible to evaluate alternative specifications. A first approach is to model default intensity as a two-state Markov process as in He, Nagel, and Song (2022). A second approach, is to assume that $d\lambda_t$ follows Cox, Ingersoll, and Ross (1985) dynamics, that is

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda t}$$

Although CIR-like dynamics would ensure that $\lambda_t > 0$, the equilibrium yield curve will not be an affine function of the state s_t. Indeed, stochastic volatility induces a second source of variation in risk premia through risk quantities. As a result, the covariance between the arbitrageurs' portfolio and

[7]See the discussion about QE effects in Krishnamurthy and Vissing-Jorgensen (2011).

bond returns turns out to be a product of two affine functions. Appendix C.4 formalizes the argument and shows why the affine conjecture fails with heteroscedastic default intensity shocks.

2.5.3 Idiosyncratic Defaults and OTC Trading

Throughout the paper, I assume that defaults are idiosyncratic. As a result, there is no risk compensation for default events, and the default probabilities are the same under both physical and martingale measure. Although the introduction of some degree of correlation across defaults would bring the model closer to reality, the key results would still go through. However, correlated default would imply a non-zero market price of default risk, which complicates the pricing of credit risk.

Finally, Friewald and Nagler (2019) point out that corporate bonds are mostly traded in over-the-counter (OTC) markets. As a result, there might be additional frictions that distinguish the market for corporate and Treasury bonds other than those highlighted in this paper. If anything, however, the premise of a few dealers managing bond inventories to provide liquidity to customers seems to provide additional support to the mechansism of concentrated risks.

3 Quantitative Analysis

This section presents a numerical exercise to illustrate the interactions between credit risk, duration risk, and bond quantities. To simplify the analysis and reduce the number of parameters in the model, I consider a stylized specification of habitat demand for asset j as a function of the price of asset j and a demand factor only, setting $\gamma^j(\tau) = \phi^j(\tau) = \rho^j(\tau) = 0$. The specification is analogous to Vayanos and Vila (2021) and Kekre et al. (2024), but it imposes strong restrictions on how risk premia respond to monetary policy and fundamental shocks. I calibrate the model targeting Treasury yields, which I obtain from Gürkaynak, Sack, and Wright (2007). I discuss data sources more in details in Section 4.

3.1 Calibration

For parsimony, I consider a single $K = 1$ demand factor β_t and I take the matrices Γ and Σ to be diagonal. Following Vayanos and Vila (2021), I assume an exponential form for the price elasticity, intercept, and slope of habitat demand such that

$$\alpha^j(\tau) = \alpha^j e^{-\delta^j_\alpha} \tag{22a}$$

$$\theta^j_1(\tau) = \theta^j_1 \left(e^{-\delta^j_\alpha \tau} - e^{-\delta^j_\theta \tau}\right) \tag{22b}$$

$$\theta^j_0(\tau) = \theta^j_0 \left(e^{-\delta^j_\alpha \tau} - e^{-\delta^j_\theta \tau}\right) \tag{22c}$$

for $\tau \leq 30$ and $\alpha^j(\tau) = \theta^j_1(\tau) = \theta^j_0(\tau) = 0$ otherwise. The equilibrium term structures of government and corporate bonds are determined by nineteen parameters. The first nine parameters characterize the dynamics of the risk factors; namely $(\kappa_r, \sigma_r, \bar{r})$ for the short rate, $(\kappa_\lambda, \sigma_\lambda, \overline{\lambda})$ for the default intensity, and $(\kappa_\beta, \sigma_\beta, \overline{\beta})$ for the demand factor. The other ten parameters control the slope $(\delta^j_\alpha, \alpha^j)$ and the intercept $(\theta^j_0, \theta^j_1, \delta^j_\theta)$ of habitat demand. The restrictions on (Γ, Σ) are akin to Gourinchas et al. (2022) and considerably simplify the estimation of the model and the interpretation of the results.

Given that only the product $\theta_1 \sigma_\beta$ matters for the equilibrium dynamics, I normalize $\sigma_r = \sigma_\beta$. I also normalize $\overline{\beta} = 0$ without loss of generality. To further reduce the number of parameters, I assume that θ^j_1, θ^j_0, and δ^j_θ are the same for both asset classes. Given that δ^j_α varies across security, however, the demand slope and the demand intercept will be different across assets. There remains 14 parameters, that is seven characterizing state dynamics $(\kappa_r, \sigma_r, \overline{r}, \kappa_\lambda, \sigma_\lambda, \overline{\lambda}, \kappa_\beta)$ and the other seven describing habitat demand $(\delta^g_\alpha, \delta^c_\alpha, \alpha^g, \alpha^c, \theta_0, \theta_1, \delta_\theta)$. Arbitrageurs' risk aversion is also a parameter to set, but it is not identified because it affects equilibrium yields only through the products $(a\alpha^j, a\theta^j_0, a\theta^j_1)$. For this reason, I set a equal to the calibration in Vayanos and Vila (2021).

Let ϑ denote the vector of model parameters. I estimate ϑ to match key unconditional moments of the Treasury term structure only. The average yield at maturity τ is

$$y^{(\tau)}_{j,t} = \frac{A_{jr}(\tau)\overline{r} + A_{j\lambda}(\tau)\overline{\lambda} + C_j(\tau)}{\tau} \tag{23}$$

and, since Γ and Σ are diagonal, the volatility of the yields is

$$\sigma\left(y^{(\tau)}_{j,t}\right) = \frac{1}{\tau}\sqrt{A_{jr}(\tau)^2 \frac{\sigma^2_r}{2\kappa_r} + A_{j\lambda}(\tau)^2 \frac{\sigma^2_\lambda}{2\kappa_\lambda} + A_{j\beta}(\tau)^2 \frac{\sigma^2_\beta}{2\kappa_\beta}} \tag{24}$$

The empirical counterparts of (23) and (24), which are the average Treasury yield and its standard deviation, respectively, are the target moments. As in Gourinchas et al. (2022), I choose ϑ to minimize the sum of the squared differences between model-implied $(\mathcal{M}_i)$ and empirical (m_i) moments. Hence

$$\widehat{\vartheta} = \arg\min L(\vartheta) \doteq \sum_i \left(\mathcal{M}_i(\vartheta) - m_i\right)^2 \tag{25}$$

I estimate ϑ from the average yields and the volatility for maturities $\tau = 1, \ldots, 20$. To speed up computations, I take the initial guess ϑ_0 to be exact same calibration in Vayanos and Vila (2021), with the exception of $\overline{r}$ and $\overline{\lambda}$, which I pick to match the level of short term yields. The value of $\overline{r}$ is set close to the historical average of the Federal Funds Rate, whereas $\overline{\lambda}$ is approximately equal the historical percentage of BBB cumulative defaults over five years.

Description	Parameter	Value	Calibration
Risk Factor Dynamics			
Short rate mean-reversion	κ_r	0.099	Own
Short rate volatility	σ_r	0.0121	Own
Short rate average	$\overline{r}$	0.015	Average Federal Funds Rate
Demand factor mean-reversion	κ_β	0.055	Vayanos and Vila (2021)
Demand factor volatility	σ_β	0.0121	Normalized to σ_r
Demand factor average	$\overline{\beta}$	0	Vayanos and Vila (2021)
Default intensity mean-reversion	κ_λ	0.049	Own
Default intensity volatility	σ_λ	0.0101	Own
Default intensity average	$\overline{\lambda}$	0.014	S&P BBB 5yr cumulative defaults
Habitat-demand Parameters			
Government elasticity decay	δ_α^g	0.299	Vayanos and Vila (2021)
Corporate elasticity decay	δ_α^c	0.297	Vayanos and Vila (2021)
Government elasticity	$a\alpha^g$	35.3	Vayanos and Vila (2021)
Corporate elasticity	$a\alpha^c$	49.846	Own
Demand intercept	$a\theta_0$	289	Vayanos and Vila (2021)
Demand factor loading	$a\theta_1$	3155.2	Vayanos and Vila (2021)
Demand loading decay	δ_θ	0.307	Vayanos and Vila (2021)

Table 1: Calibration of model parameters for the main sample of nominal yields. The sample is January 1997 to present. The calibration only targets moments of the Treasury yield curve.

Table 1 reports the parameters used in the quantitative analysis. Although the volatility of shocks to both r_t and λ_t is comparable, default intensity is significantly more persistent than the short rate ($\kappa_r > \kappa_\lambda$). As a result, the unconditional variance of default intensity is larger than the short term rate. At short maturities, since $\alpha^c > \alpha^g$, demand for risky bonds is more elastic than for Treasuries. In contrast, the exponential decay of the slope coefficient is virtually identical for both asset classes. The estimated parameter vector $\widehat{\vartheta}$ is quite close to the initial guess. Thus, I manually set the habitat demand parameters, except the product $a\alpha^c$ to match the calibration in Vayanos and Vila (2021).

3.2 Model Fit

I inspect the model fit by comparing model-implied moments and their empirical counterparts. A good fit of corporate bond yields and credit spreads is informative about whether the model captures, at least qualitatively, key moments of the data.

3.2.1 Equilibrium Characterization and Term Structure

Figure 1a plots the equilibrium term structure of Treasury bonds and defaultable bonds for maturities $\tau \in (0, 20)$. The model matches yields at short and intermediate maturities. The model-implied yields for $\tau > 15$ still provide a reasonable fit, but they are not as close to their empirical counterparts. Even though corporate yields do not play any role in the model estimation, the model-implied corporate term structure provides a good fit for maturities less than 10 years, while it deviates more at longer maturities[1]. For both corporate and government bonds, the average term structure is upward sloping.

[1]The maturity buckets of BofA ICE indices are much wider at the long end. In the estimation, I set τ equal to the midpoint of each bucket, so that the longest maturities are $\tau = 12.5$ and $\tau = 20$, but the average maturity of the index constituent may be different. At shorter maturities, the brackets are narrower, and the distribution of maturities inside each

On average, the five- and ten-year Treasury yields are 2.94% (2.96% in the data) and 3.80% (3.65% in the data), respectively. The average term spread $y_{j,t}^{(10)} - y_{j,t}^{(1)}$ is 1.95% for Treasuries and 2.44% for defaultable bonds. The average term spread on Treasury yields in the data is 1.48%.

(a) Model yields, levels $y_{j,t}^{(\tau)}$

(b) Model yields, volatility $\sigma\left(y_{j,t}^{(\tau)}\right)$

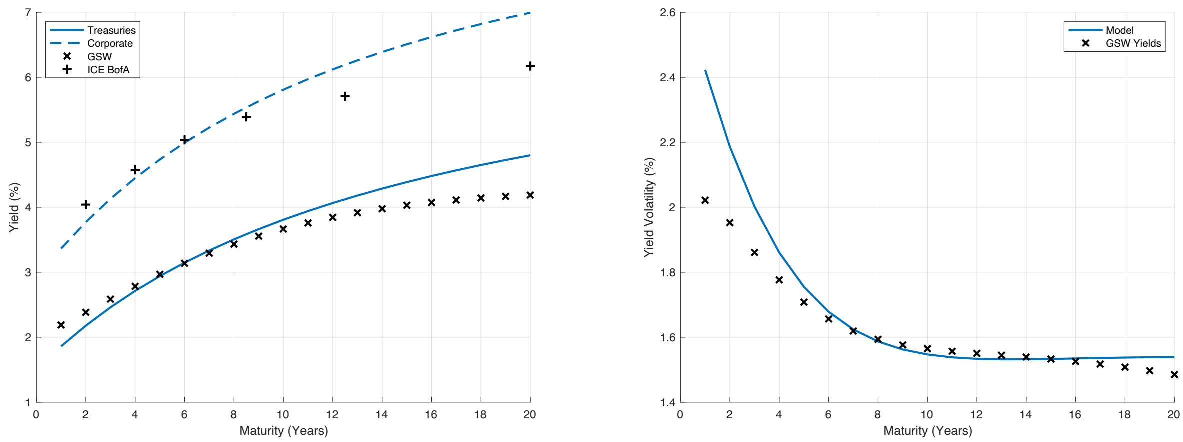


Figure 1: The left panel plots the model-implied yield curves for Treasury and corporate bonds against their empirical counterparts. The right panel plots the model-implied yield volatilities for Treasury yields against their empirical counterpart. The parameters used in the calibration are in Table 1. Treasury yields are from Gürkaynak et al. (2007), whereas bond yields are BBB effective yields from ICE BofA. The daily sample is January 1997 to present.

Figure 1b plots Treasury yield volatilities in the model and in the data. The model fits the data well at intermediate and long maturities. The unconditional volatility of the 10-year yield in the model is 1.55% (1.56% in the data). Overall, the model generates short term yields that are slightly too volatile, but the fitted yields qualitatively line up with the data. Yields volatility declines with maturity.

To analyze the propagation of short rate and default intensity throughout the Treasury yield curve, I plot the model-implied instantaneous forward rates in figure 2a and the functions $A_g'(\tau)$ in figure 2b. The model fits forward rates well at short maturities, but does not quite capture the inversion of the curve at around $\tau = 12$, where the average forward rate in the data starts to decline. Forward rates load positively onto the short rate r_t and the demand factor β_t, and negatively onto the default intensity λ_t. A shock to the short term rate has the strongest effect at short maturities, whereas demand shocks affect long term yields more. The magnitude of the response to default intensity shocks peaks at intermediate maturities, and it weakens as τ increases.

bracket is more likely to be uniform.

(a) Model instantaneous forward rates $f_{g,t}^{(\tau)}$ **(b)** Forward rate loadings on s_t

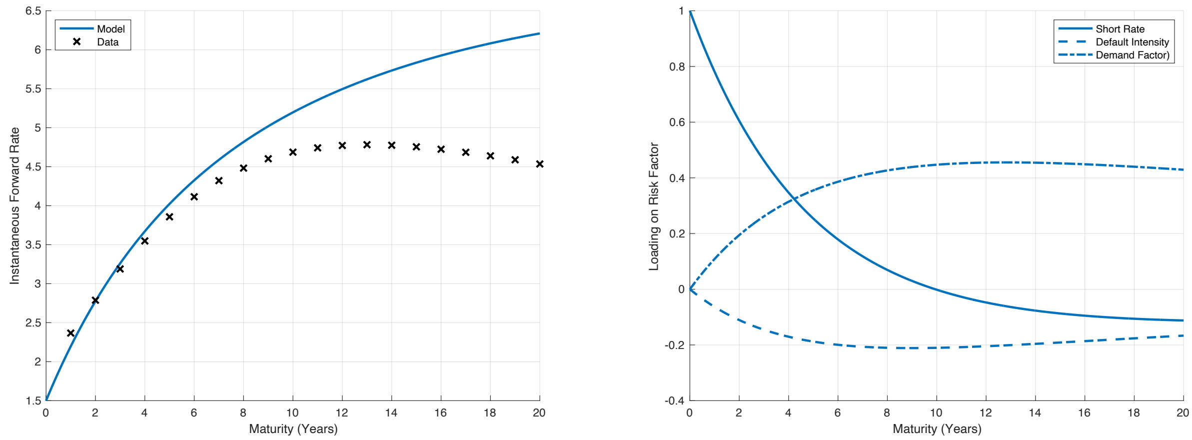


Figure 2: The left panel plots the term structure of instantaneous Treasury forward rate corporate bonds against the data. The right panel plots the loadings $A'_g(\tau)$ of instantaneous forward rates on the vector of aggregate risk factors s_t as a function of maturity. The parameters used in the calibration are in Table 1. Treasury yields are from Gürkaynak et al. (2007). The daily sample is January 1997 to present.

Overall, figure 2 validates the good model fit at short to intermediate maturities. The model also suggests that an increase in default intensity lowers forward rate, where the strongest effect is at intermediate maturities. Furthermore, it suggests that the short rate and the demand factor are relatively more important at short and long maturities, respectively. The shape of the loading on r_t, i.e. $A'_{gr}(\tau)$, is qualitatively consistent with the monotonically decreasing responses of the nominal forward rates documented in Hanson and Stein (2015) and Kekre et al. (2024).

3.3 Model Implications and Mechanisms

3.3.1 Credit Spreads

I compute model-implied credit spreads $\mathcal{S}_t^{(\tau)}$ as the difference between yields on corporate and Treasury bonds. I interpret the corporate sector as a continuum of BBB-issuer, so that I compare $\mathcal{S}_t^{(\tau)}$ with the option-adjusted spreads (OAS) from ICE BofA for BBB rated bonds at various maturities. The initial value of $\overline{\lambda}$ is chosen to match the percentage of cumulative BBB defaults within a five year horizon.

Figure 3a plots $\mathcal{S}_t^{(\tau)}$ against the time-series average OAS for BBB bonds. The model matches the average level of credit spreads accurately at short and long maturities. On average, the term structure of credit spreads is upward sloping both in the data and in the model. In the limit, as maturity tends to zero, i.e. $\tau \to 0$, credit spreads converge to the long-term average level of default intensity $\mathcal{S}_t^{(\tau)} \to \overline{\lambda}$. As a result, average yield spreads are strictly positive at zero maturity.
The shape of the term structure of credit spreads varies with the relative persistence of default intensity and the short rate, as well as with the volatility of default intensity shocks. Figure 3b shows the model-implied volatility of credit spreads. Overall, the model-implied volatility is qualitatively consistent with the data, as it monotonically declines with maturity. However, credit spreads in the

model are significantly much more volatile than in the data.

(a) Credit spreads average, level **(b)** Credit spread volatility, level

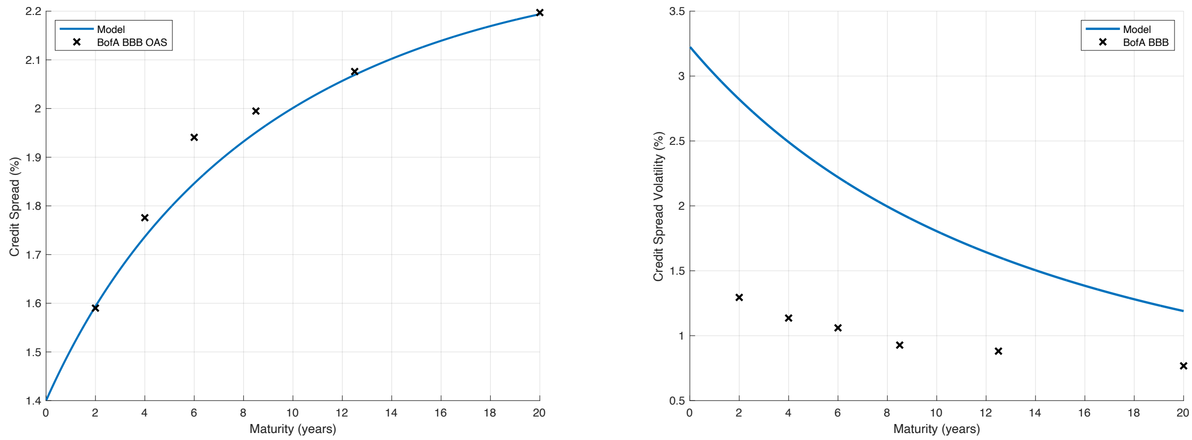


Figure 3: The left panel plots the term structure of credit spreads against their empirical counterpart. The right panel plots the volatility of credit spreads implied by the model against the data. Credit spreads are the option-adjusted spreads on BBB bonds from ICE BofA. The parameters used in the calibration are in Table 1. Treasury yields are from Gürkaynak et al. (2007). The daily sample is January 1997 to present.

I next study how the level of credit spreads loads on the short rate and the default intensity λ_t. Figure 4 plots the loadings of credit spreads on the state variables $A_{\mathcal{S}r}(\tau)$ and $A_{\mathcal{S}\lambda}(\tau)$. As expected, credit spreads are positively related to default rates. Short term spreads move one-to-one with λ_t, and the response dissipates as τ grows large. The model implies that monetary policy affects credit spreads even when expected cash flows remain unchanged. The loading is negative, meaning that an increase in the short term rate r_t lowers credit spreads.

The key mechanism is that changes in r_t affect the market price of interest rate risk. A higher r_t leads to higher yields across all markets at all maturities. Because of this, habitat investors demand more and arbitrageurs end up holding smaller net positions, lowering the equilibrium price of credit risk. As this disproportionately affects corporate yields, credit spreads decline at all maturities. The negative correlation between the short rate r_t and credit spreads is consistent with the evidence in Longstaff and Schwartz (1995) and Duffee (1999). The negative correlation between r_t and credit spreads in the data is, however, confounded by policy responses to business cycle fluctuation, whereas (3) describes how credit spreads react to an exogenous change r_t.

In this regard, Gertler and Karadi (2015) document that a contractionary monetary surprise causes an increase, rather than a decline, in various measures of credit spreads. This result is in contrast with the sign of $A_{\mathcal{S}r}(\tau)$, which implies that an exogenous shock to r_t lowers credit spreads. To explain this discrepancy, I explore the effects of an hypothetical contractionary monetary policy shock. I model this intervention as a surprise increase in the level of the short rate from $r_t = \overline{r}$ to $r_t = 1.5\overline{r}$, which is a 0.75% interest rate hike. I compute risk prices as implied by the right hand side of the arbitrageurs' first-order conditions (9) and (10), and I set λ_t and β_t equal to their long-term averages $\overline{\lambda}$ and $\overline{\beta}$.

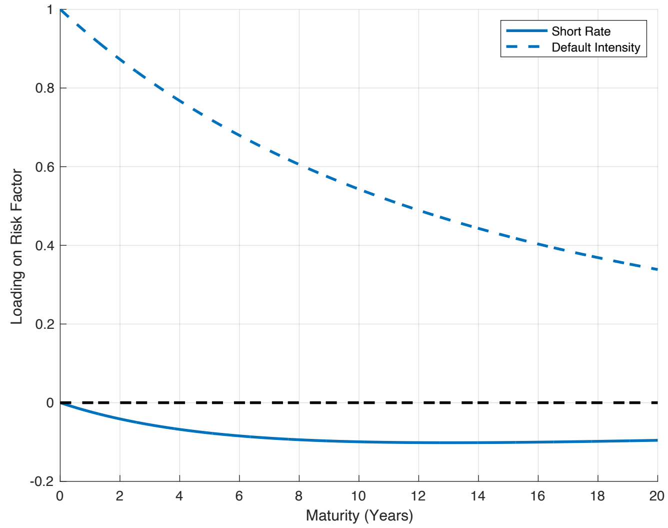


Figure 4: The figure plots the loadings $A_S(\tau)$ of credit spreads on the aggregate risk factors s_t as a function of maturity τ. The calibration is reported in Table 1.

Figure 5a plots equilibrium portfolio holdings $x_{j,t}^{(\tau)}$ as a function of maturity. The solid lines represent bond holdings at the baseline level $r_t = \bar{r}$. An increase of the short term rate from $r_t = \bar{r}$ to $r_t = 1.5\bar{r}$, keeping demand and default intensity constant, raises equilibrium yields throughout the term structure, inducing habitat investors to demand more bonds. Because of market clearing, arbitrageurs' hold now smaller net positions at all maturities, as shown by the dashed lines in figure 5a.

(a) Arbitrageurs' net positions $x_{j,t}^{(\tau)}$. **(b)** Market prices of aggregate risk.

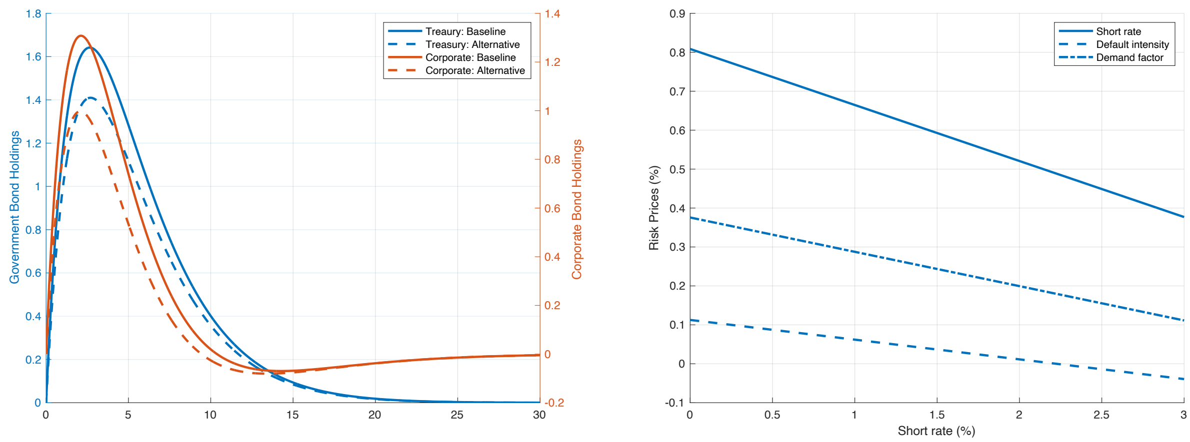


Figure 5: The left panel shows arbitrageurs' portfolio holding before and after the interest rate hike. The blue and the orange lines describe Treasury and corporate bond holdings, respectively. The right panel plots the market risk prices implied by the arbitrageurs' first-order conditions (9) and (10). The calibration is reported in Table 1. The market prices or risk are expressed as a function of r_t fixing $\lambda_t = \bar{\lambda}$ and $\beta_t = \bar{\beta} = 0$.

The price of default risk is lower because the arbitrageurs' exposure to the aggregate risk factors has declined. Given that default intensity risk mostly affects yields of defaultable bonds, the response of the corporate bonds yields weakens. In fact, the reduction in the price of default intensity risk acts in the opposite direction of the increase in r_t. Hence, the fact that monetary tightening r_t raises equilibrium habitat demand at all maturities generates a negative $A_{\mathcal{S}r}(\tau)$. Accordingly, Figure 5b plots the market prices of aggregate risk as a function of r_t fixing $\lambda_t = \overline{\lambda}$ and $\beta_t = \overline{\beta}$. There is a negative relation between r_t and the risk prices for all three state variables. An increase in the short rate induces habitat investors to save more in both bonds, reducing arbitrageurs' exposure to the aggregate risk factors and lowering risk prices.

This exercise is meant to illustrate the key mechanism of the model, since the loadings of credit spreads in risk factors depend on the parametrization of habitat demand. In particular, the sign flips whenever an increase in the short term rate induces habitat investors to demand less bonds in the aggregate. The discrepancy between Figure 4 and the results in Gertler and Karadi (2015) follows from the assumption that habitat investors only respond to prices.

3.3.2 Portfolio Rebalancing Channel

The inclusion of a second asset class to the portfolio choice problem of the arbitrageurs enriches the asset pricing implications of preferred-habitat models. In Vayanos and Vila (2021) and in the two country extensions of Gourinchas et al. (2022) and Greenwood et al. (2020), the aggregate risk factors enter the arbitrageurs' decision problem in a symmetric fashion. However, in my framework, corporate bonds default with positive probability, whereas government bonds do not. As a result, λ_t enters directly (i.e. not through market clearing) only in the first-order condition of corporate bonds $x^{(\tau)}_{j,t}$. In contrast, r_t enters directly in both the first-order conditions (9) and (10). Intuitively, when λ_t is high, the arbitrageurs require a relatively higher compensation to hold corporate bonds.

Figure 6a and 6b plot the yield loadings on the state variables $\frac{1}{\tau}A_j(\tau)$ for government and corporate bonds, respectively. While the loadings on the short rate $A_{jr}(\tau)$ and the demand shock $A_{j\beta}(\tau)$ have the same sign for both assets, the impact of default intensity λ_t on yields is asymmetric.

On the one hand, an increase in λ_t is positively related to corporate bond yields, i.e. $A_{c\lambda}(\tau) > 0$. On the other hand, the relation between Treasury yields and λ_t is negative for all maturities, i.e. $A_{g\lambda}(\tau) < 0$. It turns out that, in equilibrium, government bonds hedge against default intensity risk, and they perform well when λ_t increases.

Du et al. (2019) argue that a major challenge of structural default models is that efforts to calibrate models to observable moments have been unable to match average credit spreads levels. L. Chen et al. (2008) similarly argue that Baa–Aaa credit spreads implied by structural models of credit risk are usually significantly below historical values. L. Chen et al. (2008) then show that the puzzle can be resolved if the strong comovements in default rates and Sharpe ratios are properly accounted for.

In my model, the level of credit spreads is due to a combination of three effects. The first and more direct effect is driven by variation in the issuer credit quality λ_t. The reason is that corporate bonds default whereas government bonds do not, so that arbitrageurs require a compensation for the fraction of bonds $\lambda_t dt$ that is lost at any point in time. The second effect, which is analogous to L. Chen et al. (2008), is the correlation between the short rate and the default intensity. Furthermore, the dependence of the risk factors is even stronger under the pricing measure. The reason is that exposure to aggregate risk factors is concentrated in the arbitrageurs' portfolio, so that the equilibrium prices of credit and interest rate risk are state-dependent. The third channel is a portfolio/substitution effect, and it is captured by the opposite sign of $A_{g\lambda}(\tau)$ and $A_{c\lambda}(\tau)$. In this calibration, Treasury bonds hedge against default risk, and their price increase when λ_t goes up. This further contributes to widen credit spreads over and above the level implied by changes in the credit quality.

(a) Government bonds factor loadings

(b) Corporate bonds factor loadings

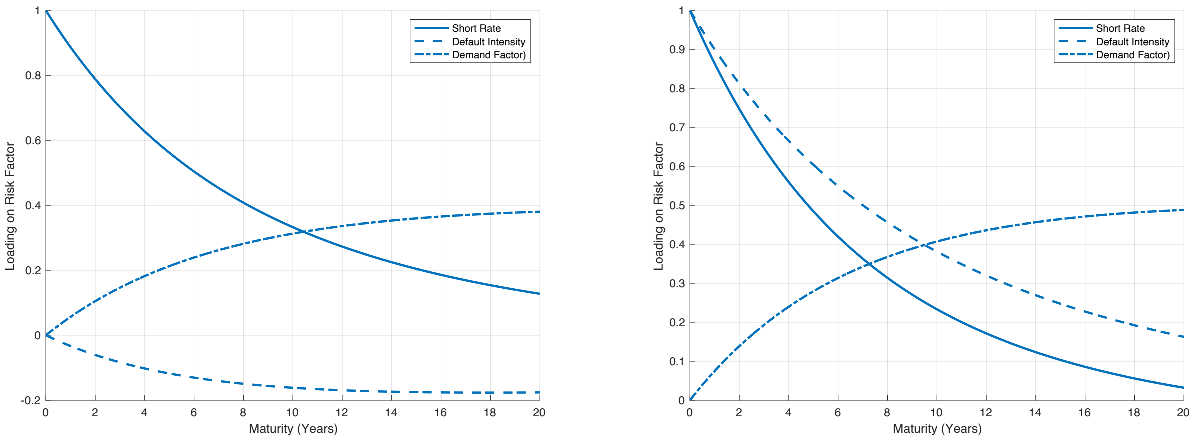
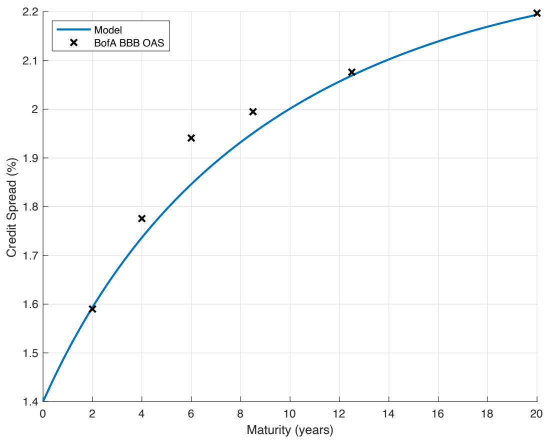


Figure 6: The figure compares the loadings of Treasury and corporate bonds on the aggregate risk factors. The loadings are the functions $A_g(\tau)$ and $A_c(\tau)$. The calibration is described in Table 1.

3.3.3 High-Yield Bonds and Rating Downgrades

A well-established fact in the corporate bond literature is that the term structure of credit spreads is upward sloping for investment grade bonds, whereas it slopes down for high-yield issuers (Jones, Mason, & Rosenfeld, 1984; Sarig & Warga, 1989). In figure 3a the term structure of credit spreads slopes up on average. The analysis so far has interpreted the corporate sector as a continuum of BBB issuers. Although default risk might be a concern, BBB-rated bonds are still investment grade securities. I now analyze the effect of a rating downgrade on credit spreads.

I model a rating downgrade as an unanticipated and permanent increase in the long term average level of default intensity $\overline{\lambda}$. I consider a moderate downgrade from BBB to BB and a severe downgrade from BBB to B. I choose the average intensity after the downgrade to match the average level of the option adjusted spreads at short maturities of the corresponding rating category. The moderate downgrade corresponds to a change from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 3.7$, whereas the severe downgrade is a change from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 6.4$. I maintain default uncertainty constant, that is the volatility of default intensity shocks σ_λ is the same before and after each downgrade. I emphasize the case of a severe downgrade to better capture the inversion of the term structure of credit spreads as the long term average $\overline{\lambda}$ rises. Figure 7a, which is identical to Figure 3a, and Figure 7b compare the term structure of credit spreads before and after the downgrade. For investment grade issuers the term structure of credit spreads is upward sloping. The average slope of the term structure of high-yield issuers is negative.



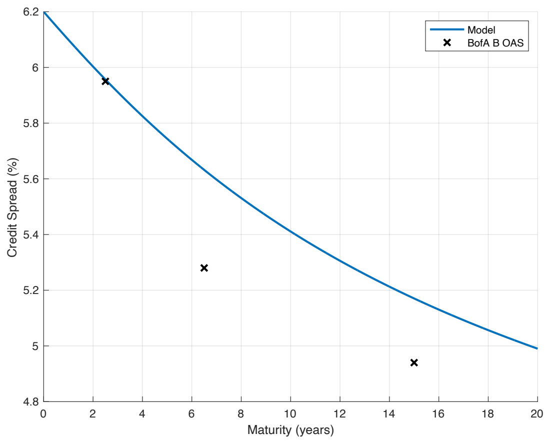


Figure 7: The figure compares the term structure of credit spreads implied by the model with the data. A credit downgrade is an unanticipated permanent increase in the long term average default intensity from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 6.4$. Option adjusted spreads (OAS) are from ICE BofA. The daily sample is January 1997 to present.

3.4 Monetary Policy Intervention

I consider two alternative monetary policy interventions to study heterogeneity in the transmission of monetary policy shocks to long rates across markets. The first intervention maps into conventional monetary policy and is modelled through an unexpected increase in the level of the short rate r_t. The second intervention I analyze is quantitative easing (QE). I initially assume that QE purchases concern government bonds only. I model QE as an unanticipated permanent decline $\Delta\theta_0^g(\tau)$ in the intercept of habitat demand of Treasury bonds. Proposition (5) states that supply shocks in the Treasury market also affect corporate yields and credit spreads by lowering risk prices. Subsequently, I consider a similar intervention where QE purchases concern corporate bonds only, which is again modeled as an unanticipated permanent decline $\Delta\theta_0^c(\tau)$ in the intercept of habitat demand of corporate bonds[2]

3.4.1 Conventional Monetary Policy

To analyze the propagation of short rate shocks throughout the Treasury yield curve, I compare the responses of instantaneous forward rates to the reaction of expected future short rates, as in Proposition (3). Figure 8a compares how instantaneous Treasury forward rates respond to a unit increase in the short term rate for three different levels of default uncertainty σ_λ, namely low ($\sigma_\lambda = 0.006$), medium ($\sigma_\lambda = 0.0101$), and high ($\sigma_\lambda = 0.0130$). Figure 8b plots the response of corporate forward rates to monetary policy shocks. In both graphs, the black dashed line represent the response of expected future short rates. As in Vayanos and Vila (2021), the model generates underreaction of forward rates to monetary policy for both asset classes. Intuitively, the extent of the overreaction is driven by arbitrageurs' risk aversion, who require a compensation to transmit monetary shocks to long term yields.

[2]The two hypotetical interventions roughly map into QE1 and QE2. QE1 included corporate bonds purchases, whereas QE2 only involved Treasuries (Krishnamurthy & Vissing-Jorgensen, 2011).

Figure 8a and 8b show that, in this particular calibration, default risk has an asymmetric impact on the strength of monetary policy transmission across asset classes. On the one hand, Treasury yields underreact less to changes in r_t when the level of default uncertainty is higher. When σ_λ is higher, monetary policy transmission to long term Treasury yields is stronger. On the other hand, at least up to intermediate maturities, the underreaction in the corporate bond market is more severe when default uncertainty is higher. When σ_λ is higher, monetary policy transmission to long term Treasury yields is weaker. This is in contrast to Vayanos and Vila (2021), where demand risk unambiguously weakens the transmission of short rate shocks to bond yields by making carry trades riskier.

(a) Treasury forward rates response to r_t. **(b)** Corporate forward rates response to r_t.

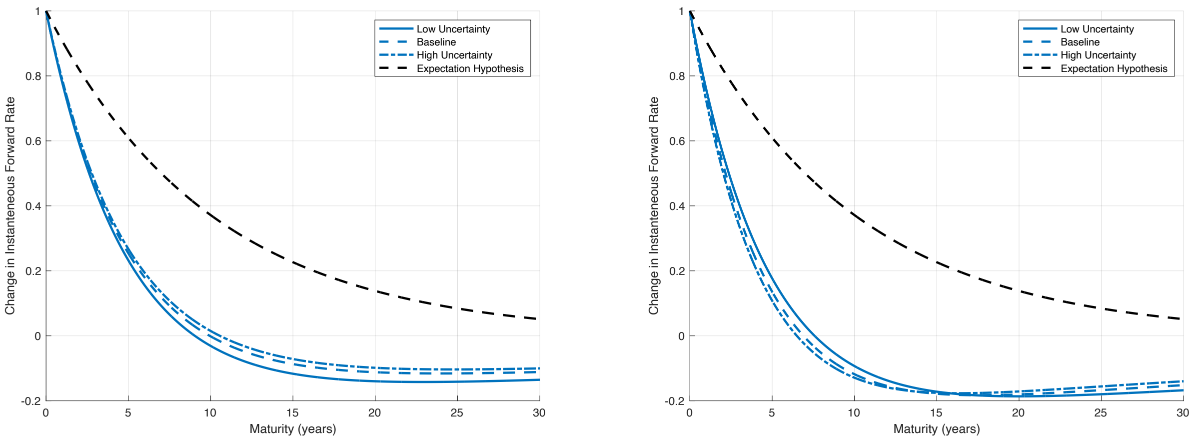


Figure 8: Underreaction of Treasury and corporate forward rates. The blue lines describe the response of forward rates to an instantaneous change in r_t for different levels of default uncertainty. The baseline uses the parameters given in Table 1. The low uncertainty case sets $\sigma_\lambda = 0.006$, whereas the high uncertainty case sets $\sigma_\lambda = 0.013$. The black dashed line plots the response of expected future short rates $\mathbb{E}_t[r_{t+\tau}]$.

The effect of higher uncertainty on corporate yields is relatively easier to interpret. Corporate bond yields load positively on default uncertainty, i.e. $A_{c\lambda}(\tau) > 0$, so that an increase in σ_λ typically makes corporate bond riskier. Monetary policy is therefore less effective in reducing the financing costs of firms when default uncertainty is high. However, the effect is the opposite in the Treasury market. In the calibration, Treasury bonds hedge against default risk, i.e. $A_{g\lambda}(\tau) < 0$ and their price increase when λ_t goes up. A higher default uncertainty makes hedging properties more valuable to arbitrageurs, lowering Treasury bond risk premia. A novel implication of this result is that the strength of monetary policy transmission across asset classes is partially determined by the interaction of the quantity of risk and the endogenous hedging properties of the assets.

3.4.2 Quantitative Easing

While Vayanos and Vila (2021) analyze the impact of QE on interest rates, their only policy target is the Treasury yield curve. However, QE works through different channels, and Treasury yields might not be the appropriate benchmark for evaluating the policy impact on the cost of capital for corporate issuers. Krishnamurthy and Vissing-Jorgensen (2011) evaluate the effects of QE interventions on the yields of different asset classes. A key implication is that the effects on particular assets depend critically on which assets are purchased. In particular, Treasury-only purchases had a disproportionate effect on Treasuries relative to corporate bonds. Furthermore, D'Amico and King (2013) show that QE interventions generate local supply effects, and that the effects are strongest for securities that are closer substitutes to Treasury bonds whose maturities coincide with the policy target.

(a) Yield curves responses to $\Delta\theta_0^g(\tau) < 0$. **(b)** Credit spread responses to $\Delta\theta_0^g(\tau) < 0$.

(c) Yield curves responses to $\Delta\theta_0^c(\tau) < 0$. **(d)** Credit spread responses to $\Delta\theta_0^c(\tau) < 0$.

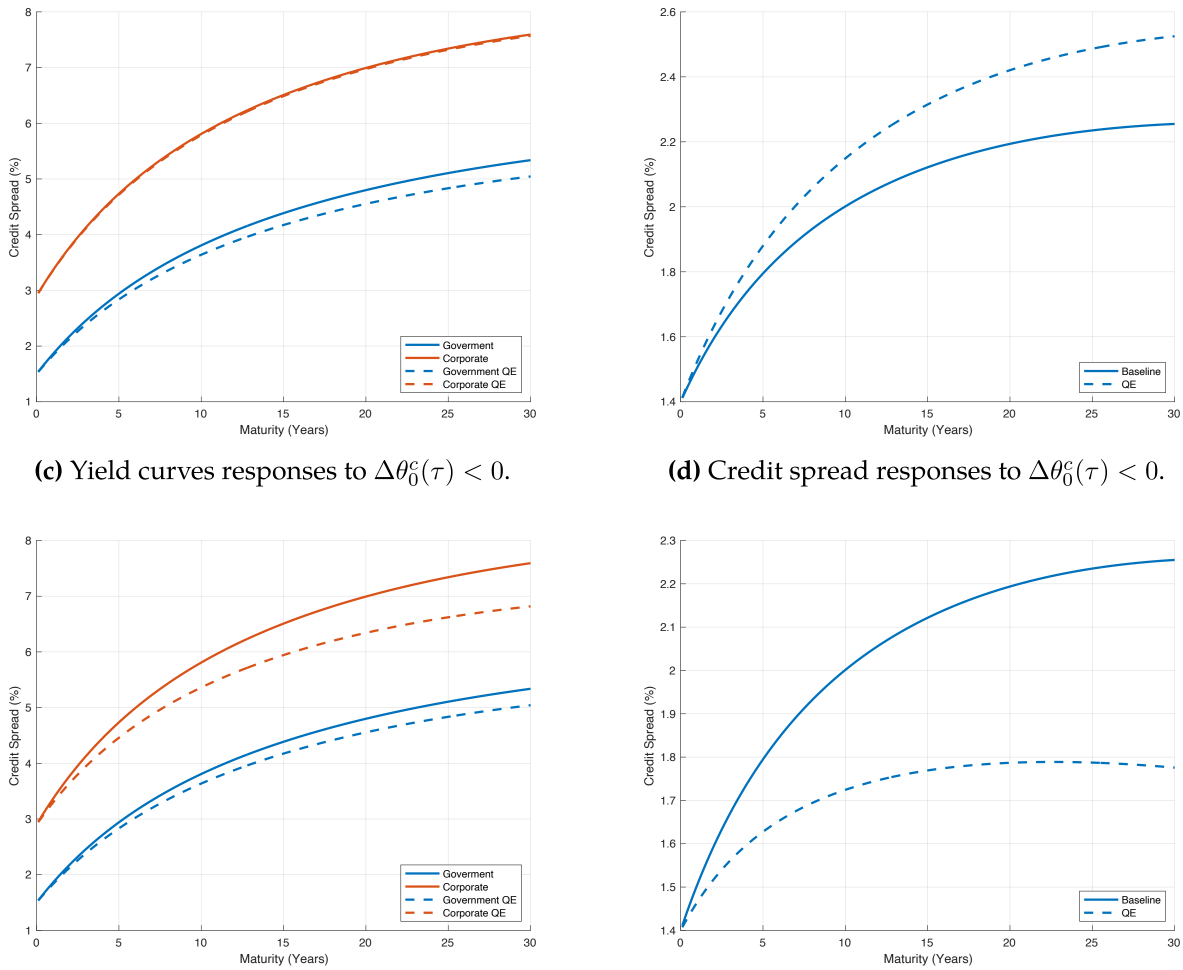


Figure 9: Impact of quantitative easing across assets. I model Treasury-only QE as an unanticipated decline in the Treasury demand intercept from $\theta_0^g = 289$ to $\theta_0^g = 260$. I model corporate-only QE as an unanticipated decline in the corporate demand intercept from $\theta_0^c = 289$ to $\theta_0^c = 260$. The parameters are in Table 1.

I study asymmetries in the effects of quantitative easing across markets by comparing credit spread responses to two alternative policy interventions. The first intervention is Treasury-only QE, and the second one is corporate-only QE. I model both interventions as an unanticipated and permanent decline in the demand intercept $\Delta\theta_0^j$. In this model, QE acts on yields and credit spreads by reducing bond residual supply. Figure 9a and Figure 9b illustrate the effect of QE purchases of government bonds only, modeled as a uniform decline in the demand intercept $\Delta\theta_0^g(\tau) < 0$ across all maturities. While the yields on Treasuries decline substantially, the impact on corporate yields is very small. As a result, when QE interventions are concentrated in the Treasury market only, credit spreads may increase. In contrast, Figure 9c and Figure 9d show that QE purchases of comparable magnitude but targeted to corporate bonds are much more effective in lowering corporate yields and credit spreads. Furthermore, a drop in $\Delta\theta_0^j(\tau) < 0$ also reduces the yields on government bonds, and the magnitude of the effect is comparable to the QE-only intervention.

In my model, the impact of QE on credit spreads is a combination of many effects. The direct effect is that a decline in habitat demand reduces the residual bond supply held by the arbitrageurs. Yields fall because risk prices typically decline. However, while also Treasuriess fall in Figure 9c and Figure 9d, corporate bond yields are barely affected by Treasury-interventions only. On the one hand, QE reduces the quantity of duration and credit risk that arbitrageurs hold in equilibrium (Greenwood & Vayanos, 2014). While Treasury-only QE is mostly about extracting duration risk, corporate-only QE also reduces the quantity of credit risk in the economy. On the other hand, purchases of government bonds also reduce the supply of safe assets and the supply of hedges against aggregate risk factors. Arbitrageurs value the hedging properties of Treasuries because they perform well in bad states of the world, when default intensity typically increases. However, a reduction in Treasury supply increases the relative scarcity of hedges and safe assets, potentially raising the equilibrium price of safety (Krishnamurthy & Vissing-Jorgensen, 2011, 2012). In summary, Figure 9 is consistent with both a portfolio rebalancing and a safety channel of quantitative easing.

4 Results

I present empirical evidence on the relation between credit and duration risk prices implied by the theoretical framework in Section 2. After briefly describing data sources, I show that the relation between credit risk premia and default risk premia varies over time and often flips sign. Second, I revisit previous evidence on determinant of credit spreads changes, emphasizing the role of duration and credit risk premia over and above economic fundamentals and aggregate uncertainty.

4.1 Data

Daily series on Treasury bond yields are from Gürkaynak et al. (2007). Daily data on corporate bond yields is from Bloomberg. ICE BofA releases corporate bond indices for different maturities and credit ratings. The indices are divided between investment grade (IG) and high-yield (HY) bonds. IG indices include six maturity buckets, $\{[1,3),[3,5)[5,7),[7,10),[10,15),[15,30)\}$ and four rating categories, AAA, AA, A, and BBB. High-yield bonds only include three maturity buckets $\{[1,5),[5,8),[8,30)\}$ and three rating categories, BB, B, and CCC. I obtain daily measures of term premia from Adrian, Crump, and Moench (2013). I measure credit risk premia using the excess bond premium (EBP) from Gilchrist and Zakrajšek (2012). Macro controls (industrial production and CPI) and financial indicators (VIX, Federal funds rate) are from the St. Louis Fed. Stock excess return are from Kenneth French's data library, and expected default frequencies (EDF) are from Moody's.

4.2 Treasury Term Premium and Credit Risk

Figure 10 plots monthly series of the GZ spread and the excess bond premium together with the 5-year Treasury term premium. In the first part of the sample, there is no clear pattern in the co-movements of credit premium and term premium. In contrast, bond risk premia are strongly positively correlated during the Great Financial Crises. Between 2007 and 2009, the 5-year term premium and the excess bond premium rise together. After the GFC, however, the credit premium and the term premium move in opposite directions. The negative correlation is particularly striking in march 2020 at the onset of the Covid-19 pandemic, when a sudden increase in the excess bond premium comes along with contemporaneous reduction in the term premium (He, Nagel, & Song, 2022).

(a) GZ spread and TP 5Y. **(b)** EBP and TP 5Y.

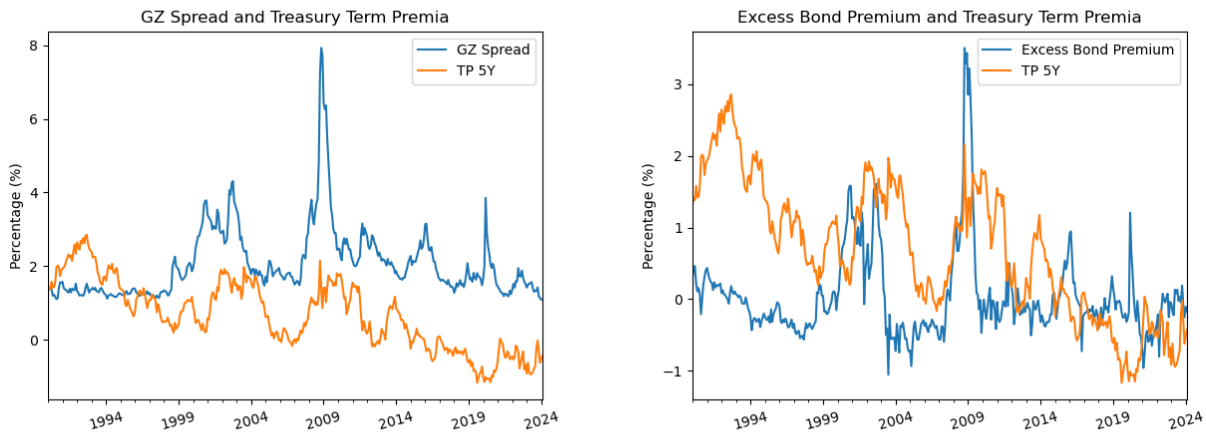


Figure 10: The figure plots term premia and credit spreads. The left panel compares the co-movement of 3–5 year OAS for BBB-rated issuers to five year term premia. The right panel compares the co-movement of 10–15 year OAS for BBB-rated issuers to the five year term premia. OAS are from ICE BofA, whereas term premia are from Adrian et al. (2013). The daily sample is from January 1997 to present.

To further investigate time variation in the relation between the term premium and and credit risk premium, I consider the linear regression model

$$y_t = \beta_0 + \beta_1 \cdot \mathrm{TP}_t^{(10)} + \beta_2 \cdot \delta_t \mathrm{TP}_t^{(10)} + \delta_t + \gamma \cdot \mathrm{Controls}_t + \varepsilon_t \tag{26}$$

where TP_t is the 10-year term premium. I compare estimates using three dependent variables, namely the BAA-AAA spread, the GZ spread, and the excess bond premium. I interact the periods dummies δ_t with TP_t to allow for time-variation in the slope coefficient β_2. I consider three periods, that is (i) the Great Financial Crisis from July 2007 to June 2009 (ii) the zero lower bound (ZLB) period from July 2009 to December 2014 (iii) and the post GFC period without Covid from January 2010 to March 2020. The set of controls includes both financial variables (stock market excess returns and VIX) and macro variables (change in log CPI ad change in log industrial production). The coefficients of interest β_1 and β_2 describe co-movements of credit and duration risk premia. I report estimates in Table 2.

The dependent variables in Columns (1) and (2) are the BBB-AAA yield spread and the GZ spread, respectively. The relation between term premia and these two measures of credit spreads is at most weakly positive but not statistically significant. However, the relation between the term premium and the excess bond premium, which captures the non-default component of credit spreads, is positive and statistically significant, as shown in Column (3). Estimates in Column (4) document substantial time-variation in this relation. First, the positive co-movements between these two variables are much stronger during the Great Financial Crisis. Second, the relation is significantly weaker and even flips sign in the ZLB period, confirming the visual intuition from Figure 10. Except attenuating the slope coefficient on the interaction between $\mathbb{1}\{\text{GFC}\}$ and term premia, controlling for financial indicators and macro variables has little effect on the estimates.

	BAA–AAA	**GZ Spread**	**EBP**	**EBP**	**EBP**	**EBP**
TP 10Y	0.02	0.09	0.10**	0.09**	0.13***	0.12***
	(0.03)	(0.08)	(0.04)	(0.04)	(0.03)	(0.03)
$\mathbb{1}\{\text{GFC}\} \times$ TP 10Y				1.20***	0.66***	0.62***
				(0.19)	(0.19)	(0.18)
$\mathbb{1}\{\text{Post}\} \times$ TP 10Y				-0.08	-0.04	-0.03
				(0.06)	(0.07)	(0.07)
$\mathbb{1}\{\text{ZLB}\} \times$ TP 10Y				-0.03	-0.15**	-0.12*
				(0.06)	(0.06)	(0.07)
$\mathbb{1}\{\text{GFC}\}$				-0.68*	-0.17	-0.15
				(0.37)	(0.38)	(0.36)
$\mathbb{1}\{\text{Post}\}$				0.17	0.37***	0.29***
				(0.11)	(0.10)	(0.10)
$\mathbb{1}\{\text{ZLB}\}$				-0.08	-0.16*	-0.16**
				(0.11)	(0.08)	(0.08)
R^e_t					0.00	-0.00
					(0.01)	(0.01)
VIX					0.04***	0.03***
					(0.01)	(0.01)
Δip_t						-0.06*
						(0.04)
Δcpi_t						-0.22***
						(0.07)
Intercept	0.93***	1.95***	-0.08	-0.17**	-0.98***	-0.81***
	(0.03)	(0.08)	(0.05)	(0.08)	(0.15)	(0.14)
R^2	0.00	0.01	0.04	0.44	0.60	0.62
Adj. R^2	0.00	0.01	0.04	0.43	0.59	0.61
N	410	410	410	410	410	409

Table 2: OLS estimates of specification (26). The monthly sample is January 1990 to February 2024. TP 10Y is the 10 year term premium from Adrian et al. (2013). EBP is the excess bond premium from Gilchrist and Zakrajšek (2012). The period dummy $\mathbb{1}\{\text{GFC}\}$ takes the value of one from July 2007 to June 2009 and zero otherwise. The period dummy $\mathbb{1}\{\text{Post}\}$ takes the value of one from January 2010 to March 2020 and zero otherwise. The period dummy $\mathbb{1}\{\text{ZLB}\}$ takes the value of one from July 2009 to December 2014 and zero otherwise. Newey and West (1987) standard errors in parentheses. * $p < .1$, ** $p < .05$, *** $p < .01$.

4.3 Return Predictability Regressions

The arbitrageurs' first-order conditions imply that Treasury yields vary with default risk. Accordingly, measures of credit risk premia should predict Treasury excess bond returns. I explore this theoretical prediction in the data using Treasury nominal yields from Gürkaynak et al. (2007). I use the excess bond premium of Gilchrist and Zakrajšek (2012) as a proxy for the market price of credit risk. I restrict the monthly sample is 1990 to 2020, but I explore how estimates change when including the Covid-19 pandemic[1] in Appendix D.

I regress Treasury excess returns for maturities $\tau \in \{2, \dots, 30\}$ on the excess bond premium. I estimate the monthly linear regression model

$$rx^{(\tau)}_{g,t|t+h} = \beta^{(\tau)}_0 + \beta^{(\tau)}_1 \cdot \text{EBP}_t + x_t + \varepsilon_{t+h} \tag{27}$$

where $rx^{(\tau)}_{g,t|t+h}$ is the h-period excess return on a zero-coupon Treasury bond with maturity τ and x_t is a vector of controls. I construct Treasury holding period returns as

$$rx^{(\tau)}_{g,t|t+h} = \frac{P^{(\tau-h)}_{g,t+h}}{P^{(\tau)}_{g,t}} - y^{(h)}_t$$

where $y^{(h)}_t$ is the risk-free yield for maturity h. To account for the overlapping forecast horizons, I compute Hodrick (1992) standard errors[2]. I first report estimates for a one-year holding period returns on nominal bonds without including any control. I then repeat the same exercise controlling for the 10-year Treasury term premium (Adrian et al., 2013) and the VIX.

The left panel of Figure 14 plots the baseline estimates of the linear regression (27) and associated 90% and 95% confidence intervals. The excess bond premium significantly predicts one-year Treasury bond excess returns for longer maturities. In contrast, the excess bond premium is positively related to excess returns on Treasury bonds at shorter maturities, but the estimates are barely significant.

[1]Point estimates are virtually identical, but confidence intervals are slightly wider.

[2]The asymptotic covariance matrix of the coefficients is $\Theta = R_x(0)^{-1}\mathfrak{G}R_x(0)^{-1}$ where $\mathfrak{G} = \sum_{j=-k+1}^{k-1} R_u(j)R_x(j)$. I estimate of $R_u(j)$ and $R_x(j)$ as $\hat{R}_x(j) = \sum_{t=j+1}^{T} x_t' x_{t-j}$ and $\hat{R}_u(j) = \sum_{t=j+1}^{T} \hat{u}_t' \hat{u}_{t-j}$ where x_t is the vector of predictors.

(a) Treasury excess returns and EBP. **(b)** Treasury excess returns, EBP and controls

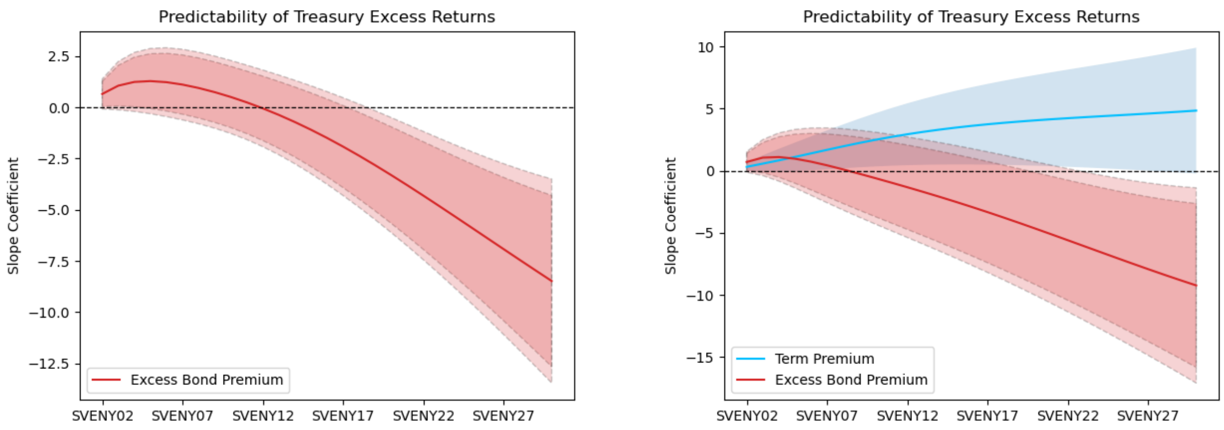


Figure 11: Parameter estimates of regression (27). The left panel presents regressions of nominal bond excess returns on the excess bond premium (EBP) (Gilchrist & Zakrajšek, 2012). The right panel presents regressions of nominal bond excess returns on the excess bond premium controlling for the Treasury term premium (Adrian et al., 2013) and the VIX. Shaded areas represent 90% and 95% confidence intervals constructed using Hodrick (1992) standard errors. The monthly sample is January 1990 to January 2020.

The right panel of Figure 14 shows that coefficient estimates remain statistically significant when controlling for the Treasury term premium and the VIX. As expected, an increase in the Treasury term premium positively predicts Treasury excess returns, whereas an increase in the excess bond premia is negatively associated to excess bond returns at longer maturities. In summary, I find that an increase in the excess bond premium predicts excess returns on Treasury bonds over and above the Treasury term premium and the VIX. The effect is negative for long maturities and positive for shorter maturities.

Combined with the evidence presented in Table (2), these results provide support to the prediction that Treasury yields vary with the price of credit risk.

4.4 Credit Spread Changes and Bond Risk premia

I next explore whether movements in bond risk premia explain changes in credit spreads over and above economic fundamentals. To this purpose, I build on Collin-Dufresne et al. (2001) and consider regressions of the form

$$\Delta \text{OAS}_t^{(\tau),r} = \beta_0 + \beta_1 \cdot \Delta \text{TP}_t^{(5)} + \beta_2 \cdot \Delta \text{EBP}_t + \beta_3 \cdot \Delta \text{EDF}_t + \beta_4 \cdot \Delta \text{HPW}_t + \gamma \cdot \text{Controls}_t + \epsilon_t \quad (28)$$

where $\text{OAS}_t^{(\tau),r}$ is the ICE BofA option adjusted spread for rating category r and maturity τ. The first two regressors capture changes in term premia and credit risk premia. Changes in the Moody's EDF proxy for the expected default component of credit spreads. The additional controls include the federal funds rate, VIX and industrial production growth to proxy for aggregate uncertainty and macroeconomic conditions. I control for intermediary distress using the noise measure of Hu, Pan, and Wang (2013). The inclusion of these controls is motivated by structural models of default

(Collin-Dufresne & Goldstein, 2001; He, Khorrami, & Song, 2022), although the analysis is at the aggregate level. Equation (28) does not seek to establish causality, but to help assess if variations in credit spreads are driven by changes in fundamentals, changes in risk premia, or a combination of both. Table 3 report coefficient estimates for BBB-rated bonds of 1–3 year and 15+ year maturities.

Estimates in Columns (1) through (2) show that there is a weak negative relation between term premia and short term credit spreads. Conversely, the relation is negative and statistically significant for long maturity bonds, suggesting that changes in duration risk premia have heterogeneous effects on corporate and Treasury bonds. Across all specifications, there is a clear positive and significant relation between changes in the excess bond premium and changes in OAS. The magnitude of the coefficient is larger at shorter maturities, and declines for bonds with longer maturities. The slope coefficient on changes in the non-default component of credit spreads remains significant when controlling for aggregate uncertainty and expected default. Controlling for intermediary distress raises the regression R^2, and estimates for β_1 and β_2 become strongly significant.

Changes in risk premia alone generate R^2 of roughly 40%. In contrast, estimates of β_3 are not statistically significant when the excess bond premium and the VIX are also included in the regression. Further, including changes in expected defaults does not improve the regression R^2 in both specification (5) and increase to 60% in specification (10). Regressions R^2 are roughly 57% in specification (6) and specification 67% in specification (12). While the sign of the coefficients is consistent with Collin-Dufresne and Goldstein (2001) and He, Khorrami, and Song (2022), the regression R^2 are higher.

	Δ BBB OAS – 1–3 year						Δ BBB OAS – 15+ year					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
ΔTP 5Y	0.02	0.05	-0.29**	-0.35**	-0.32*	-0.34**	-0.17	-0.15	-0.31***	-0.31***	-0.28***	-0.30***
	(0.46)	(0.28)	(0.14)	(0.16)	(0.18)	(0.17)	(0.24)	(0.12)	(0.08)	(0.09)	(0.08)	(0.08)
ΔEBP		0.97***	0.71***	0.71***	0.63***	0.63***		0.61***	0.48***	0.48***	0.40***	0.39***
		(0.30)	(0.18)	(0.17)	(0.17)	(0.17)		(0.13)	(0.10)	(0.10)	(0.09)	(0.09)
ΔHPW			0.18***	0.17***	0.15**	0.15***			0.09***	0.09***	0.06***	0.07***
			(0.07)	(0.06)	(0.06)	(0.06)			(0.03)	(0.03)	(0.02)	(0.02)
ΔFFR				-0.16	-0.17	-0.17				-0.00	-0.02	-0.02
				(0.11)	(0.12)	(0.11)				(0.04)	(0.04)	(0.04)
ΔVIX					0.02***	0.02***					0.02***	0.02***
					(0.00)	(0.00)					(0.00)	(0.00)
ΔEDF						1.80						0.09
						(5.81)						(2.69)
Δip_t						-4.37						-4.22*
						(3.01)						(2.32)
Intercept	-0.00	-0.01	-0.01	-0.01	-0.01	-0.01	0.01	0.00	0.00	0.00	0.00	0.00
	(0.04)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
R^2	0.00	0.41	0.53	0.54	0.56	0.57	0.02	0.49	0.57	0.57	0.65	0.67
Adj. R^2	-0.01	0.40	0.52	0.53	0.55	0.55	0.01	0.49	0.56	0.56	0.64	0.65
N	191	191	191	191	191	191	191	191	191	191	191	191

Table 3: OLS estimates of the linear regression model (28). The dependent variable is the change in the BBB OAS for maturities short and long maturities, respectively. ΔTP 5Y is the monthly change in the 5-year term premium from Adrian et al. (2013). ΔEBP is the monthly change in the excess bond premium from Gilchrist and Zakrajšek (2012). ΔEDF is the monthly change in Moody's expected default frequencies. ΔHPW is the monthly change in the noise measure of Hu et al. (2013). The monthly sample is September 1999 to January 2016. Newey and West (1987) standard errors are reported in parentheses.

An increase in the short term rate is negatively correlated with changes in credit spreads across all the specifications. The negative relation between the level of Treasury yields and credit spreads is well documented in the literature (e.g. Duffee (1998)). In this case, the negative correlation likely also captures policy responses to business cycle fluctuations. Overall, Table 3 suggest that variation in risk premia matters for changes in credit spreads over and above economic fundamentals and aggregate uncertainty, consistent with the insight from Du et al. (2019) and L. Chen et al. (2008).

Table 4 report the counterparts of Columns (6) and (12) for all available categories. Estimates show that similar patterns hold across rating categories and maturities. For investment grade bonds, the slope coefficient β_1 increase monotonically with credit ratings. Indeed, estimates of β_1 for AAA bonds are positive and significant across all maturities, with the exception of the 10–15 year bucket. The coefficient on changes in the excess bond premium β_2 is almost always positive and statistically significant. Within each maturity bucket, β_2 declines with the credit rating, suggesting that a change in bond risk premia has a stronger effect on BBB credit spreads. Further, the coefficient on expected default frequencies is almost always statistically indistinguishable from zero, and even negative in some cases. Regression R^2 are around 50% for BBB-rated bonds, but are much lower for safe AAA bonds.

		ΔTP 5Y		ΔEBP		ΔEDF 1M			
Maturity	Rating	β_1	SE(β_1)	β_2	SE(β_2)	β_3	SE(β_3)	R^2	Adj. R^2
Investment Grade									
1–3 year	AAA	0.13	0.07	0.15	0.06	-14.73	7.02	0.37	0.34
	AA	-0.09	0.09	0.20	0.05	-10.42	5.16	0.57	0.55
	A	-0.10	0.12	0.39	0.11	-15.06	8.14	0.51	0.49
	BBB	-0.34	0.17	0.63	0.17	1.80	5.81	0.57	0.55
3–5 year	AAA	0.11	0.05	0.22	0.06	-8.29	4.84	0.48	0.46
	AA	-0.07	0.07	0.24	0.07	-8.34	4.77	0.51	0.49
	A	-0.15	0.09	0.34	0.07	-6.81	4.32	0.56	0.55
	BBB	-0.42	0.20	0.57	0.13	4.60	4.43	0.61	0.60
5–7 year	AAA	0.06	0.10	0.15	0.08	4.53	9.75	0.22	0.19
	AA	-0.07	0.06	0.25	0.06	-5.75	3.21	0.56	0.55
	A	-0.17	0.10	0.34	0.08	-6.48	4.30	0.54	0.52
	BBB	-0.51	0.20	0.52	0.12	3.57	4.90	0.61	0.59
7–10 year	AAA	0.10	0.08	0.18	0.06	-8.92	5.08	0.36	0.34
	AA	-0.14	0.06	0.18	0.06	-2.92	3.60	0.50	0.48
	A	-0.20	0.08	0.32	0.08	-4.97	4.07	0.55	0.53
	BBB	-0.43	0.15	0.48	0.11	3.13	3.84	0.64	0.63
10–15 year	AAA	-0.05	0.14	0.16	0.06	-6.28	4.74	0.08	0.04
	AA	-0.18	0.06	0.16	0.06	0.72	3.89	0.20	0.17
	A	-0.32	0.11	0.25	0.08	1.28	2.45	0.37	0.35
	BBB	-0.35	0.14	0.43	0.08	3.57	3.97	0.59	0.57
15+ year	AAA	0.10	0.12	0.15	0.06	-9.42	5.63	0.34	0.31
	AA	-0.17	0.07	0.20	0.06	-0.65	2.75	0.44	0.42
	A	-0.16	0.06	0.27	0.06	-0.46	2.57	0.58	0.56
	BBB	-0.30	0.08	0.39	0.09	0.09	2.69	0.67	0.65
High Yield									
1–5 year	BB	-0.83	0.42	1.17	0.33	3.32	21.62	0.48	0.46
	B	-0.53	0.28	0.97	0.20	11.74	9.56	0.61	0.60
	CCC	0.51	1.10	1.98	0.73	-1.34	34.69	0.35	0.32
5–8 year	BB	-0.84	0.24	0.74	0.14	9.07	6.23	0.63	0.61
	B	-1.03	0.30	1.10	0.22	8.94	8.86	0.64	0.63
	CCC	-2.17	0.84	1.84	0.40	17.34	27.99	0.53	0.52
8+ year	BB	-0.88	0.22	0.64	0.11	10.94	6.68	0.56	0.54
	B	-1.02	0.23	1.20	0.26	16.00	9.28	0.55	0.53
	CCC	-1.67	0.69	1.51	0.48	10.09	31.12	0.39	0.37

Table 4: Estimates of regression 28 across rating category and maturities. Standard errors are computed as in Newey and West (1987). The monthly sample is September 1999 to January 2016.

As far as high-yield spreads are concerned, there is still a monotonic relation between β_2 and credit rating within maturity buckets. Except for CCC-rated bonds with maturity 1–5 years, the relation between term premia and credit spreads is negative. An increase in the market price of duration lowers credit spreads. After controlling for changes in risk premia, the slope coefficient on expected default frequencies β_3 is statistically insignificant.

5 Conclusion

Motivated by the insights that the variation in credit spreads is driven by time-varying risk premia rather than default probabilities and that intermediary-based factors explain a substantial fraction of the common variation in credit spreads, I study a model of the term structure of Treasury and corporate yields in which corporate and Treasury bonds are jointly priced by the same marginal investor. I integrate elements from the literature on credit risk valuation in a preferred-habitat context where asset prices are jointly determined by the pricing kernel of arbitrageurs that trade in both the Treasury and the corporate bond markets. I use my model to study (i) the interaction between credit and interest rate risk, (ii) the determinants of credit spreads, and (iii) how monetary policy interventions propagate throughout the term structure of credit spreads. I discipline the model to provide qualitative answers through a calibration exercise targeting empirical moments of the Treasury yield curve.

The propositions in the two sector model, as well as the calibration exercise, hint at a very strong dependence between credit risk and interest rate risk. In a context in which arbitrageurs are pricing both corporate and Treasury bonds, this dependence is strengthened under the risk-neutral measure. Portfolio rebalancing effects have the potential to enrich asset pricing implications of habitat models and to shed more light on monetary policy transmission in a setting where assets are asymmetrically exposed to risk factors. The fact that risk prices of interest rate and credit risk are interconnected might explain some of the credit spread puzzles documented in the literature.

Nevertheless, the quantitative analysis reveals some limitations, which provide clear guidance onto where future efforts should directed. First, the implication that exogenous shocks to the short rate reduce credit spread is at odds with the literature. Future work is devoted to present empirical evidence of this mechanism and to understand how the model can match the data. Second, the specification of habitat demand lacks a solid microfoundation along two dimensions. On the one hand, it is unclear why habitat investors only respond to the price of a single maturity. On the other hand, fundamental news only affects habitat demand through prices, preventing these investors to react to fundamental shocks in the first place. In this regard, a better microfoundation of habitat demand is central to link habitat investors to key players in the corporate bond market as well as to generate realistic responses of risk premia to the aggregate risk factors. Third, the model suggests that intermediary inventories play a role in determining bond excess return. This results should be connected more tightly to the literature on intermediary asset pricing. Fourth, most of the asset pricing implications of the two sector model have not been tested yet. Improvements to the calibration procedure and a more thorough empirical analysis are necessary to better assess whether the model captures key features of the data.

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Appendices

A Model Extensions

A.1 Endogenous Habitat Demand

I provide an alternative formulation of habitat demand to emphasize the limitation of downward sloping demand curve that are typically assume in the preferred-habitat literature. The framework presented in Section 2 imposes strong restrictions on the behavior of habitat investors and on the dynamics of the state variables. Some of these restrictions are alleviated by assuming that habitat demand varies directly with r_t and λ_t. However, demand specifications as in Vayanos and Vila (2021) and Gourinchas et al. (2022) are less suitable for the corporate bond market.

In the corporate bond market, investors are more likely to respond to economic fundamental directly, and not only through their effect on prices. In the data, a deterioration in the credit quality of the corporate sector is commonly associated to selling pressure from pension funds and institutional investors (Bao, O'Hara, & Zhou, 2018; He, Khorrami, & Song, 2022). Regulatory constraints generate a similar pattern also for insurance companies (Ellul, Jotikasthira, & Lundblad, 2011; Koijen & Yogo, 2015). Downward sloping demand curves counterfactually imply that habitat demand increase after an adverse economic shock lowers the price of defaultable assets. Furthermore, the habitat demand specification cannot properly account for how investors account for risk. A deterioration in credit quality lowers prices, and so does an increase in volatility of the shocks. Finally, the specification of the state dynamics implies that λ_t might become negative with positive probability.

A.1.1 Environment

The environment is as in Section 2. Time is continuous and runs from zero to infinity. Financial markets consist of zero-coupon Treasury and corporate bonds.

Arbitrageurs Let $j \in \{g, c\}$ index government and corporate bonds, respectively. Arbitrageurs have mean-variance preferences over instantaneous changes in wealth

$$\max_{\{x_{j,t}^{(\tau)}\}_{j\in\{g,c\},\tau\in\{0,\infty\}}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\mathrm{ar}_t(dW_t)\right] \tag{29}$$

The arbitrageurs' budget constraint is

$$dW_t = \left(W_t - \int_0^\infty \sum_j x_{j,t}^{(\tau)} d\tau\right) r_t dt + \int_0^\infty x_{g,t}^{(\tau)} \frac{dP_{g,t}^{(\tau)}}{P_{g,t}^{(\tau)}} d\tau + \int_0^\infty x_{c,t}^{(\tau)} \left(\frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}} - \lambda_t dt\right) d\tau \tag{30}$$

Arbitrageurs hold both corporate bonds and Treasury bonds at all maturities.

Hedgers (Habitat Investors) A first major deviation from the literature is the modelling of market segmentation and the preferences of habitat investors. I replace the habitat investors of Vayanos and Vila (2021) with short-lived hedgers in the spirit of Kondor and Vayanos (2019). There are two classes of short-lived hedgers: Treasury hedgers and corporate bond hedgers. Treasury hedgers solve

$$\max_{z_{g,t}^{(\tau)}} \mathbb{E}_t\left(dW_{g,t}\right) - \frac{a^g}{2}\mathbb{V}\text{ar}_t\left(dW_{g,t}\right) \tag{31}$$

where a^g is the coefficient of absolute risk aversion. The budget constraint is

$$dW_{g,t} = \left(W_{g,t} - z_{g,t}^{(\tau)}\right)\left[1+\theta_g(\tau)\right] r_t dt + z_{g,t}^{(\tau)} \frac{dP_{g,t}^{(\tau)}}{P_{g,t}^{(\tau)}} + \sqrt{\lambda_t}\cdot\delta_g(\tau) dB_t \tag{32}$$

Corporate bond hedgers solve an analogous problem, but with potentially different risk aversion a^c. Their budget constraint is

$$dW_{c,t} = \left(W_{c,t} - z_{c,t}^{(\tau)}\right)\left[1+\theta_c(\tau)\right] r_t dt + z_{c,t}^{(\tau)} \left(\frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}} - \lambda_t dt\right) + \sqrt{\lambda_t}\cdot\delta_c(\tau) dB_t \tag{33}$$

The first term in both Equation (32) and (33) is the return from investing in the short-term rate, augmented with a wedge $\theta_j(\tau)$. The wedge $\theta_j(\tau)$ reflect non-pecuniary benefits or a liquidity premium that accrue from investing in the risk-free asset. A positive value of $\theta_j(\tau) > 0$ means that the risk-free asset is more convenient. The level of the liquidity premium is allowed to vary with the short-term rate (Krishnamurthy & Li, 2023; Nagel, 2016). The second term is the return from investing in either Treasury bonds or corporate bonds with maturity τ, adjusted for expected defaults λ_t. The third term captures the sensitivity of hedgers' wealth to fundamental shocks, along the lines of Kondor and Vayanos (2019). The vector $\delta_j^{(\tau)} = [\delta_{jr}^{(\tau)}, \delta_{j\lambda}^{(\tau)}]$ captures heterogeneity in the sensitivity to shocks across investors and across maturities. For example, $\delta_j^{(\tau)} dB_t$ could capture the return on a benchmark that hedgers are tracking (Pavlova & Sikorskaya, 2022). Similarly, the term could also capture existing positions of short-lived hedgers whose value potentially varies with innovations to the short term rate and to default intensity (Jansen, 2023; Kekre et al., 2024). In the special cases that $\theta_j(\tau) = 0$, $a^g = a^c$, and $\delta_j^{(\tau)} = [0, 0]$, the problem of hedgers and arbitrageurs is essentially identical.

Risk factor dynamics I consider a simplified economy with two risk factors. The aggregate risk factors are the short rate r_t, the default intensity λ_t. The 2×1 vector $s_t \doteq (r_t, \lambda_t)$ follows the process

$$ds_t = -\Gamma(s_t - \bar{s})dt + \sqrt{\lambda_t}\Sigma dB_t \tag{34}$$

where $\bar{s}$ is a 2×1 vector of long-term averages and $dB_t = (dB_{r,t}, dB_{\lambda,t})^T$ is a 2×1 vector of independent Brownian motions. The specification is analogous to Duffie, Pedersen, and Singleton (2003), with the difference that default intensity drives stochastic volatility. The state dynamics in Equation (34) feature stochastic volatility in both shocks to the short rate and default intensity. For both state variables, the volatility of the innovations increases with λ_t.

Market Clearing Bond markets for $j \in \{g, c\}$ clear at each maturity τ

$$x_{g,t}^{(\tau)} + z_{g,t}^{(\tau)} = 0 \quad : \quad \tau \in (0, \infty)$$
$$x_{c,t}^{(\tau)} + z_{c,t}^{(\tau)} = 0 \quad : \quad \tau \in (0, \infty)$$

at each point in time.

A.1.2 Equilibrium

An equilibrium is a collection of prices $\left\{P_{g,t}^{(\tau)}, P_{c,t}^{(\tau)}\right\}_{\tau \in (0,\infty)}$ and quantities $\left\{x_{g,t}^{(\tau)}, x_{c,t}^{(\tau)}, z_{g,t}^{(\tau)}, z_{c,t}^{(\tau)}\right\}_{\tau \in (0,\infty)}$ such that (i) arbitrageurs optimize (ii) Treasury and corporate hedgers optimize and (iii) markets clear. I make the simplifying assumption that the risk factor are independent, that is matrices Γ and Σ are diagonal. This is without loss of generality, but it considerably lightens the notation. The dynamics of the state variables are then given by

$$dr_t = \kappa_r(\bar{r} - r_t)dt + \sqrt{\lambda_t}\sigma_r dB_{r,t}$$
$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sqrt{\lambda_t}\sigma_\lambda dB_{\lambda,t}$$

A.1.3 Equilibrium with Arbitrageurs

I conjecture that corporate and Treasury yields are affine functions of r_t and λ_t, that is

$$P_{j,t}^{(\tau)} = e^{-[A_{jr}(\tau)r_t + A_{j\lambda}(\tau)\lambda_t + C(\tau)]} \tag{35}$$

for $j \in \{g, c\}$. Under the exponentially-affine conjecture, instantaneous expected returns are

$$\frac{dP_{j,t}^{(\tau)}}{P_{j,t}^{(\tau)}} = \mu_{j,t}^{(\tau)} dt - A_{jr}(\tau)\sqrt{\lambda_t}\sigma_r dB_{r,t} - A_{j\lambda}(\tau)\sqrt{\lambda_t}\sigma_\lambda dB_{\lambda,t} \tag{36}$$

where

$$\begin{aligned} \mu_{j,t}^{(\tau)} = A_{jr}'(\tau)r_t + A_{j\lambda}'(\tau)\lambda_t + C_j'(\tau) + A_{jr}(\tau)\kappa_r(r_t - \overline{r}) \\ + A_{j\lambda}(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) + \frac{1}{2}A_{jr}(\tau)^2\sigma_r^2\lambda_t + \frac{1}{2}A_{j\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \end{aligned} \tag{37}$$

In contrast to the case with homoscedastic shocks, expected returns increase with λ_t also because higher values of default intensity imply higher volatilities of the shocks. Further, instantaneous returns inherit homsocedasticity from the state variables. Substituting (37) into the arbitrageurs' budget constraint yields the following Proposition.

Proposition 6 (Arbitrageurs' First Order Condition with Stochastic Volatility)**.** *The first order conditions for $x_{g,t}^{(\tau)}$ and $x_{c,t}^{(\tau)}$ are*

$$\mu_{g,t} - r_t = -\sigma_r\sqrt{\lambda_t}A_{gr}(\tau)\eta_{r,t} - \sigma_\lambda\sqrt{\lambda_t}A_{g\lambda}(\tau)\eta_{\lambda,t} \tag{38}$$
$$\mu_{c,t} - r_t = \lambda_t - \sigma_r\sqrt{\lambda_t}A_{cr}(\tau)\eta_{r,t} - \sigma_\lambda\sqrt{\lambda_t}A_{c\lambda}(\tau)\eta_{\lambda,t} \tag{39}$$

where the market prices of interest rate and default intensity risk $\eta_{s,t}$ for $s \in \{r, \lambda\}$ are

$$\eta_{s,t} = -a\sigma_s \sqrt{\lambda_t} \left[\sum_j \int_0^\infty x_{j,t}^{(\tau)} A_{js}(\tau) d\tau \right] \tag{40}$$

Proposition 6 is the counterpart of Equations (9) and (10) with stochastic volatility. Expected excess returns equal the sum of risk exposures times the market prices of risk. The main difference is that risk prices and expected excess returns would vary with λ_t even when $\eta_{s,t}$ is constant. As before, risk prices are pinned down through market clearing, so I solve for hedgers' demand next. With heteroscedastic shocks, there is no guarantee that the right hand side of equations (38) and equations (39) maintains an affine structure in the state variables. Intuitively, stochastic volatility introduces a second sources of variation in risk premia on top of the fluctuations induced by time-varying risk prices.

I characterize Treasury and corporate bond hedgers' demand in the next Proposition.

Proposition 7 (Hedgers' Demand)**.** *Treasury hedgers' demand is*

$$z_{g,t}^{(\tau)} = \frac{1}{\lambda_t} \left(\Psi_{gr}(\tau) r_t + \Psi_{g\lambda}(\tau) \lambda_t + \Psi_{g0}(\tau) \right) + \Omega_g(\tau) \tag{41}$$

whereas the demand of corporate bond hedgers is given by

$$z_{c,t}^{(\tau)} = \frac{1}{\lambda_t} \left(\Psi_{cr}(\tau) r_t + \Psi_{c\lambda}(\tau) \lambda_t + \Psi_{c0}(\tau) \right) + \Omega_g(\tau) \tag{42}$$

where $\Psi_{jr}(\tau)$, $\Psi_{jr}(\tau)$, $\Psi_{j0}(\tau)$ and $\Omega_j(\tau)$ are given in Appendix C.2.8.

The demand of both Treasury hedgers' and corporate bond hedgers is a non-linear function of the state variable. The sensitivity of demand to shocks declines with λ_t. When λ_t increases, the conditional volatility of the shocks is larger and uncertainty increases. As a result, hedgers reduce their demand for risky bonds. Hedgers' sensitivities to the risk factors $\Psi_{js}(\tau)$ are endogenous function of maturity that depend on the loadings of prices on the state variables. The sign of the loadings describes how hedgers' demand varies in response to shocks to the risk factors. In equilibrium, these coefficients can be positive or negative depending on the wedges $\theta_j(\tau)$ and $\delta_g(\tau)$. In Appendix A.3, I show that similar demand curves arise when hedgers are modelled as delegated fund managers subject to a benchmark or as long term investors that seek to minimize duration mismatch between assets and liabilities.

Market clearing in both the Treasury and corporate bond sector require

$$x_{g,t}^{(\tau)} = -z_{g,t}^{(\tau)} = -\frac{1}{\lambda_t} \left(\Psi_{gr}(\tau) r_t + \Psi_{g\lambda}(\tau) \lambda_t + \Psi_{g0}(\tau) \right) - \Omega_g(\tau)$$

and

$$x_{c,t}^{(\tau)} = -z_{c,t}^{(\tau)} = -\frac{1}{\lambda_t} \left(\Psi_{cr}(\tau) r_t + \Psi_{c\lambda}(\tau) \lambda_t + \Psi_{c0}(\tau) \right) - \Omega_c(\tau)$$

Substituting $x_{c,t}^{(\tau)}$ and $x_{g,t}^{(\tau)}$ into (40) gives

$$\eta_{s,t} = -a\sigma_s\sqrt{\lambda_t}\sum_j \int_0^\infty \left(\frac{1}{\lambda_t}\left(\Psi_{jr}(\tau)r_t + \Psi_{j\lambda}(\tau)\lambda_t + \Psi_{j0}(\tau)\right) + \Omega_j(\tau)\right) A_{js}(\tau)d\tau \tag{43}$$

As in Section 2, the market prices of credit and duration risk are state dependent. However, risk prices are no longer an affine function of s_t. Indeed, fluctuations of default intensity have two different effects on $\eta_{s,t}$. The first channel is the same as above. A shock to default intensity loads to changes in the composition of arbitrageurs' portfolio. The second channel is driven by the conditional variance of the shocks. When λ_t is large, hedgers' demand is less sensitive to r_t and λ_t, which dampens fluctuations in the quantities that arbitrageurs must absorb for markets to clear. Risk prices also depend on $\theta_j(\tau)$ and $\delta_{js}(\tau)$ through the coefficients $\Psi_{jr}(\tau)$ and $\Omega_j(\tau)$.

The result revisits the well-known challenge of CIR-style models that risk premiums are proportional to factor volatilities, so it is only through time-varying volatilities that risk premiums can vary (Dai & Singleton, 2002; Duffee, 2002). In equation (43), the market prices of interest rate and credit risk are a non-linear functions of the state variables, and they vary even when shocks are conditionally homoscedastic. Substituting equation (43) into the arbitrageurs' first-order conditions gives

$$\begin{aligned}\mu_{g,t} - r_t = -a\sigma_r^2 A_{gr}(\tau)\sum_j \int_0^\infty &\left[\Psi_{jr}(\tau)r_t + \left(\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right)\lambda_t + \Psi_{j0}(\tau)\right] A_{jr}(\tau)d\tau \\ - a\sigma_\lambda^2 A_{g\lambda}(\tau)\sum_j \int_0^\infty &\left[\Psi_{jr}(\tau)r_t + \left(\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right)\lambda_t + \Psi_{j0}(\tau)\right] A_{j\lambda}(\tau)d\tau\end{aligned}$$

and

$$\begin{aligned}\mu_{c,t} - r_t = \lambda_t - a\sigma_r^2 A_{cr}(\tau)\sum_j \int_0^\infty &\left[\Psi_{jr}(\tau)r_t + \left(\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right)\lambda_t + \Psi_{j0}(\tau)\right] A_{jr}(\tau)d\tau \\ - a\sigma_\lambda^2 A_{c\lambda}(\tau)\sum_j \int_0^\infty &\left[\Psi_{jr}(\tau)r_t + \left(\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right)\lambda_t + \Psi_{j0}(\tau)\right] A_{j\lambda}(\tau)d\tau\end{aligned}$$

Because the two equations must hold for all values of r_t and λ_t. The next Proposition describes the system of ODEs that fully characterizes the unknown functions $A_{jr}(\tau)$, $A_{j\lambda}(\tau)$, and $C_j(\tau)$.

Proposition 8. *The functions $A_{jr}(\tau)$ and $A_{j\lambda}(\tau)$, solve the system of quadratic ODEs*

$$\begin{aligned}A'_{gr}(\tau) &= 1 - A_{gr}(\tau)\kappa_r^* - A_{g\lambda}(\tau)\kappa_{r\lambda}^* \\ A'_{cr}(\tau) &= 1 - A_{cr}(\tau)\kappa_r^* - A_{c\lambda}(\tau)\kappa_{r\lambda}^* \\ A'_{g\lambda}(\tau) &= -A_{gr}(\tau)\kappa_{\lambda r}^* - A_{g\lambda}(\tau)\kappa_\lambda^* - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 \\ A'_{c\lambda}(\tau) &= 1 - A_{c\lambda}(\tau)\kappa_\lambda^* - A_{cr}(\tau)\kappa_{\lambda r}^* - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2\end{aligned}$$

with the boundary conditions $A_{gr}(0) = 0$, $A_{cr}(0) = 0$, $A_{g\lambda}(0) = 0$, and $A_{c\lambda}(0) = 0$. Further, the functions

$C_j(\tau)$ are given by

$$C_g(\tau) = \kappa_r^* \bar{r}^* \int_0^\tau A_{gr}(u)du + \kappa_\lambda^* \bar{\lambda}^* \int_0^\tau A_{g\lambda}(u)du$$
$$C_c(\tau) = \kappa_r^* \bar{r}^* \int_0^\tau A_{cr}(u)du + \kappa_\lambda^* \bar{\lambda}^* \int_0^\tau A_{c\lambda}(u)du$$

The coefficients κ_r^*, κ_λ^* , $\kappa_{r\lambda}^*$, $\kappa_{r\lambda}^*$, $\kappa_r^* \bar{r}^*$, and $\kappa_\lambda^* \bar{\lambda}^*$ are given in Appendix C.2.9.

Proposition (8) describes a system of quadratic integral ODEs in the four unknown functions $A_{jr}(\tau)$ and $A_{j\lambda}(\tau)$. The quadratic terms in the third and fourth equation arise because of stochastic volatility. The coefficients are integrals of the functions $A_{j,s}(\tau)$ over the entire domain $\tau \in (0, \infty)$. I solve the system numerically, as described in Appendix B. To further isolate fluctuations in risk premia driven by time-varying risk prices, I solve the same model but with homoscedastic shocks in Appendix **??**.

To find the model-implied risk neutral dynamics, I write $dB_t^{\mathbb{Q}} - \eta_t dt = dB_t$ where η_t stacks $\eta_{r,t}$ and $\eta_{\lambda,t}$. Because Σ and Γ are diagonal, it follows that

$$dr_t = -\kappa_r(\bar{r} - r_t)dt + a\sigma_r^2 \sum_j \int_0^\infty \left(\Psi_{jr}(\tau) r_t + \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] \lambda_t + \Psi_{j0}(\tau)\right) A_{jr}(\tau) d\tau dt + \sqrt{\lambda_t} \Sigma dB_{r,t}^{\mathbb{Q}}$$
$$d\lambda_t = -\kappa_\lambda(\bar{\lambda} - \lambda_t)dt + a\sigma_\lambda^2 \sum_j \int_0^\infty \left(\Psi_{jr}(\tau) r_t + \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] \lambda_t + \Psi_{j0}(\tau)\right) A_{j\lambda}(\tau) d\tau dt + \sqrt{\lambda_t} \Sigma dB_{r,t}^{\mathbb{Q}}$$

The same insights as in Section 2 hold. State-dependent risk premia imply that the risk-neutral dynamics of the short rate depend on the level of default intensity, and vice versa. A given demand system for Treasury and corporate bond hedgers produces testable restrictions on the co-movements of credit and duration risk premia. Analogously, co-movements in term premia and default risk premia in the data are informative about hedgers' preferences.

A.1.4 Equilibrium without Arbitrageurs

A useful benchmark is the equilibrium without arbitrageurs. In this case, market clearing reduces to

$$z_{j,t}^{(\tau)} = 0 \quad : \quad j \in \{g, c\} \qquad \tau \in (0, \infty)$$

where the demand curves of Treasury and corporate bond hedgers are

$$z_{g,t}^{(\tau)} = \frac{\mu_{g,t}^{(\tau)} - (1 + \theta_g(\tau)) r_t}{a^g \lambda_t \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2\right]} + \frac{\delta_{r,g}^{(\tau)} A_{gr}(\tau)\sigma_r + \delta_{\lambda,g}^{(\tau)} A_{g\lambda}(\tau)\sigma_\lambda}{A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2}$$

and

$$z_{c,t}^{(\tau)} = \frac{\mu_{c,t}^{(\tau)} - \lambda_t - (1 + \theta_c(\tau)) r_t}{a^c \lambda_t \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2\right]} + \frac{\delta_{r,c}^{(\tau)} A_{cr}(\tau)\sigma_r + \delta_{\lambda,c}^{(\tau)} A_{c\lambda}(\tau)\sigma_\lambda}{A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2}$$

Equivalently, since $[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2] > 0$ and $\lambda_t > 0$

$$\mu_{j,t}^{(\tau)} - \lambda_t - (1+\theta_j(\tau))r_t = \delta_{r,j}^{(\tau)} A_{jr}(\tau)\sigma_r + \delta_{\lambda,c}^{(\tau)} A_{j\lambda}(\tau)\sigma_\lambda$$

Under the exponentially-affine conjecture, expected bond returns are given in (37). Matching coefficients produces two independent systems of ordinary differential equations, one for each asset class. In the Treasury market, I obtain

$$A'_{gr}(\tau) + A_{gr}(\tau)\kappa_r - (1+\theta_g(\tau)) = 0 \tag{44a}$$

$$A'_{g\lambda}(\tau) + A_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 = 0 \tag{44b}$$

whereas in the corporate bond market

$$A'_{cr}(\tau) + A_{cr}(\tau)\kappa_r - (1+\theta_c(\tau)) = 0 \tag{45a}$$

$$A'_{c\lambda}(\tau) + A_{c\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2 - 1 = 0 \tag{45b}$$

While the exact solution for $A_{jr}(\tau)$ depends on the functional form $\theta_j(\tau)$, the equilibrium is characterized by two independent systems of ODEs. Because bond markets are segmented, corporate and Treasury bonds are priced separately and credit spreads are (partially) disconnected from differences in expected defaults. Further, equation (44b) is satisfied when $A_{g\lambda}(\tau) = 0$, and Treasury yields only have a one-factor structure even when λ_t induces stochastic volatility in the short-rate process. Conversely, corporate yields load on both r_t and λ_t. When Treasury and corporate bond markets are fully segmented, shocks to default intensity do not propagate to the Treasury market.

A.2 Discussion of Empirical Predictions

I show that when habitat investors have mean variance preferences as in Greenwood et al. (2018), segmentation is not sufficient to generate variation in risk premiums over time. Failure of the expectations hypothesis requires residual supply to vary over time. However, when all agents have mean variance preferences, who holds the asset in equilibrium depends only on the relative risk tolerance.

A.2.1 Expectations Hypothesis and Forward Rate Underreaction

When the expectations hypothesis (EH) of the term structure holds, bond risk premia are constant and do not vary with the state variables. If that is the case, forward rates move one-to-one with expected future short rates (Vayanos & Vila, 2021). Instantaneous forward rates are again given by

$$f_{j,t}^{(\tau)} = A'_{jr}(\tau)r_t + A'_{j\lambda}(\tau)\lambda_t + C'_j(\tau)$$

so that

$$\frac{\partial f_{j,t}^{(\tau)}}{\partial r_t} = A'_{jr}(\tau) = 1 - A_{jr}(\tau)\kappa_r^* - A_{j\lambda}(\tau)\kappa_{r\lambda}^*$$

In the next Proposition, I characterize the solution of $A_{gr}(\tau)$ and $A_{cr}(\tau)$ when the expectation hypothesis holds. Indirectly, the result describes how expected future short rates vary with r_t.

Proposition 9 (Expectations Hypothesis and Risk Premiums)**.** *If $\delta_j(\tau) = 0$ and $\theta_j(\tau) = 0$, the expectations hypothesis holds and bond risk premiums are constant. If $\theta_j(\tau) \neq 0$, then bond risk premiums vary with the short rate r_t. If $\delta_j(\tau) \neq 0$, then bond risk premiums vary with the level of default intensity λ_t. When the expectations hypothesis holds, $A_{gr}(\tau) = A_{cr}(\tau)$, where*

$$A_{jr}(\tau) = \frac{1 - e^{-\kappa_r \tau}}{\kappa_r}$$

Proposition (9) implies that bond risk premia vary with the state variables only when the hedgers' demand loads differently on to r_t and λ_t as compared to arbitrageurs. In contrast, when the expectations hypothesis holds, risk premiums are constant and do not vary with shocks to the short rate and to default intensity. In that case, the response of instantaneous forward rates to short rate shocks is

$$\frac{\partial f_{j,t}^{(\tau)}}{\partial r_t} = A_{jr}(\tau) = e^{-\kappa_r \tau}$$

When $\theta_j(\tau) \neq 0$, however, hedgers' demand is less elastic to short rate shocks than arbitrageurs' demand. Because of this, shocks to the short rate have an impact on risk prices, and forward rates do not move one-to-one with expected future short rates. In the habitat tradition, the assumption of downward sloping demand curves implies that forward rates always underreact to monetary shocks (see Proposition 2 in Vayanos and Vila (2021). In this framework, what matters is whether hedgers are more or less responsive to the short term rate as arbitrageurs. If hedgers are more elastic, $\theta(\tau) > 0$ and forward rate overreact. If hedgers are less elastic, $\theta(\tau) < 0$ and forward rate underreact.

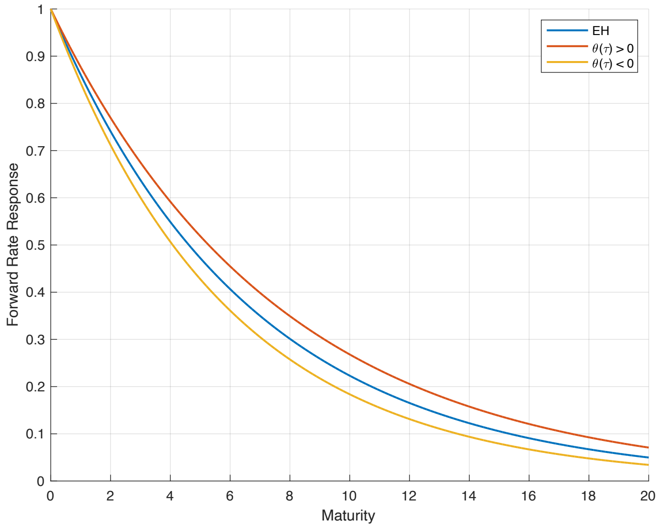

Figure 12: I plot $A'_{gr}(\tau)$ for three different cases. In case 1, the expectations hypothesis (EH) holds. In case 2, $\theta(\tau) > 0$ and forward rates overreact to shocks to the short rate. In case 3, $\theta(\tau) > 0$ and forward rates underreact to shocks to the short rate. The model parameters are set to $\kappa_r = 0.15$, $\kappa_\lambda = 0.12$, $\sigma_r = \sigma_\lambda = 0.12$, $a = 2$, and $a^g = a^c = 40$. In case 1, $\theta^g(\tau) = \theta^c(\tau) = 0$. In case 2, $\theta^g(\tau) = \theta^c(\tau) = 0.08$. In case 3, $\theta^g(\tau) = \theta^c(\tau) = -0.08$.

To illustrate this idea, Figure 12 plots $A'_{gr}(\tau)$ for different specifications of $\theta^j(\tau)$. I specify the $\theta(\tau) > 0$ and $\theta(\tau) < 0$ to be constant for simplicity, so that it is easy to characterize the direction of the effect. When these functions vary with τ, what matters is the average response across all the hedgers, which involves integrals of $\theta^j(\tau)$ and $A_{js}(\tau)$. Further, the calibration used in Figure 12 sets a very high value for the hedgers' risk aversion. This is necessary when whenever the functions $\theta^j(\tau)$ are constant, as mean-variance preference imply that, as $\tau \to 0$, demand elasticity explodes.

When the expectation hypothesis holds, $A_{gr}(\tau) = A_{cr}(\tau)$ and credit spreads do not depend on the level of the short rate r_t. This is because the loading of corporate and Treasury yields on r_t is the same, so that movements in the short term rate generate *parallel* movements of the two yield curves. When risk premiums are state-dependent, however, responses to the short rate differ across asset classes.

A.2.2 Excess Bond Premium, Credit Spreads, and Monetary Policy

Proposition 9 show that the whether forward rate overreacts or underreacts to short term innovations depends on $\theta^j(\tau)$. More generally, the sign of the response of risk prices to short term rate and default intensity depends on whether hedgers' demand elasticity to r_t and λ_t is higher or lower than the arbitrageurs.

I illustrate the mechanism by computing yield loadings on risk factors for the cases in which hedgers face no frictions (EH), for the cases in which they are more elastic to both r_t and λ_t, the case in which they are more elastic to λ_t but not r_t, and vice versa. Figure 13 plots $\frac{1}{\tau}A_{js}(\tau)$ as a function of maturity. The blue dashed line corresponds to the case in which the expectations hypothesis holds, and Proposition 9 holds. The loadings on r_t and λ_t varies depending on whether hedgers are more or less sensitive to innovation to the state variables.

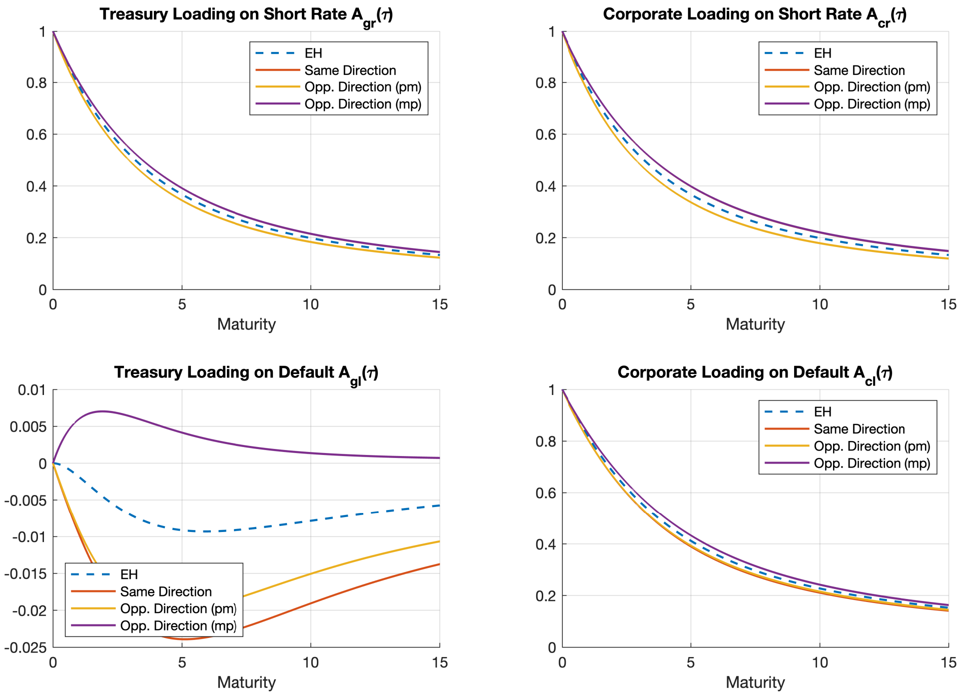

Figure 13: Corporate and Treasury bond loading on risk factors r_t and λ_t.

In the baseline, $A_{g\lambda}(\tau) < 0$ and Treasury bonds hedge against shocks to default intensity. As a result, an increase in λ_t lowers Treasury yields. When hedgers are less sensitive to default intensity shocks than arbitrageurs, however, Treasury bonds load *positively* on default intensity. Intuitively, a deterioration of credit quality induces arbitrageurs and hedgers to sell defaultable securities and purchase government bonds, reducing Treasury yields. However, when their demand is imperfectly elastic, are less elastic, hedgers' may trade more aggressively in the corporate bond market, so that Treasury prices must go down to restore equilibrium.

A.3 Microfoundation of Habitat Demand with Square-root Dynamics

The specification of habitat demand in Section 2 has the convenient properties that (i) the martingale dynamics of the state vector can be characteriyed analytically and (ii) that, in equilibrium, yields are affine functions of the state variables. Although this particular specification of habit demand is common in the literature (Costain et al., 2022; Gourinchas et al., 2022; Vayanos & Vila, 2021), it suffers from three main shortcomings. First, there is no guarantee that default intensity is strictly positive. Second, habitat agents only respond to prices, but not to the economic fundamentals s_t. As a result, habitat investors act as liquidity providers and trade favorably to the arbitrageurs, meaning that they want to buy when intermediaries wish to sell. Third, the specification of habitat demand lacks a clear microfoundation.

Building on these insights, I revisit the segmentation model without demand shocks in Section **??** to incorporate two additional elements. First, I assume that the dynamics of the risk factors r_t and λ_t

are given by the square-root process

$$dr_t = \kappa_r(\overline{r} - r_t)dt + \sigma_r\sqrt{\lambda_t}dB_{r,t}$$
$$d\lambda_t = \kappa_\lambda(\overline{\lambda} - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t}$$

where the shocks to the short rate and to default intensity $dB_{r,t}$ and $dB_{\lambda,t}$ are independent. These dynamics ensure that $\lambda_t > 0$ almost surely and imply that both r_t and λ_t are heteroscedastic. However, I make the simplifying assumption that the heteroscedasticity is entirely driven by λ_t. Technical conditions on the existence of strong solutions to the SDE are given in Duffie and Kan (1996). The only restriction required for the existence of a unique strong solution is $\kappa_\lambda\overline{\lambda} > \frac{\sigma_\lambda^2}{2}$, which I assume to be true.

The main challenge with square-root dynamics is that the stochastic volatility introduces a second source of variation in risk premia. While in Section 2 risk premia only vary with the quantities absorbed by the arbitrageurs, square-root dynamics entail that the quantity of risk also varies over time. Without any further change, this alone would imply that the covariance between the pricing kernel and bond returns is a product of two affine functions. Unfortunately, such a result would rule out an equilibrium yield curve that is still affine in the aggregate risk factors,

Second, I specify habitat demand such that habitat investors respond to economic fundamentals and consider the risk profile of the securities they are allowed to invest in. As before, habitat investors, indexed by $\tau \in (0, \infty)$, are uniformly distributed across maturities and only hold corporate bonds with a specific maturity τ. Investors with habitat τ at time t hold a position

$$z_t^{(\tau)} = \frac{\alpha(\tau)}{\lambda_t}\left[\mu_t^{(\tau)} - r_t - \lambda_t\right] + \frac{\alpha(\tau)}{\lambda_t}\beta_t^{(\tau)} + \theta(\tau) \tag{46}$$

in the bond with maturity τ and hold no other bonds. Specification (46) has three components. The first term is the standard mean-variance demand, adjusted to account for the fact that a fraction of bonds $\lambda_t dt$ is defaulting at any instant. The second term collects demand shocks that affect expected returns, such as investment managers' skills, regulatory costs, or information frictions. The third term captures the inelastic component of demand, which can be driven by compensation schemes linear in a benchmark (Pavlova & Sikorskaya, 2022) or by duration matching of liabilities.

Henceforth, I consider the case with $K = 0$ demand factors. Accordingly, $\alpha(\tau)$ is given by

$$\alpha(\tau) = \frac{1}{a^H\left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right]} \tag{47}$$

where a^H denotes the risk-aversion of the habitat investors. As in the standard mean-variance framework, the sensitivity of demand to expected return is inversely proportional to fundamental risk, i.e. σ_r and σ_λ, and to risk-aversion a^H. The function (47) can be exactly microfounded by assuming that habitat investors maximize an instantaneous mean-variance objective, but are only allowed to trade bonds with maturity τ or invest at the risk-free rate. Specification (46) is very similar to the habitat demand in Vayanos and Vila (2021). It features a price-elastic term, demand shocks, and a demand intercept. However, there are important differences that make equation (46) more suitable for risky assets such as corporate bonds.

First, habitat demand depends on the risk-return profile of the asset. Not only does $z_t^{(\tau)}$ respond to expected returns, but it also accounts for the volatility of bond returns, as captured by the the denominator of $\alpha(\tau)$. Furthermore, habitat investors become less price elastic when default intensity is higher. Second, $z_t^{(\tau)}$ directly responds to the economic fundamentals r_t and λ_t, and not only through their effect on prices. Third, the sensitivity of demand to expected returns $\frac{\alpha(\tau)}{\lambda_t}$ is endogenous and varies over time. The fact that λ_t appears in the denominator is a convenient property that allows me to solve for an equilibrium affine yield curve even in presence of stochastic volatility. I next describe two different ways to justify Equations (46) and (47), giving a concrete identity to habitat investors.

A.3.1 Habitat Investors: Mutual Funds & ETFs

I interpret habitat investors as delegated portfolio managers, whose compensation is linked to a bond benchmark. The benchmark varies across maturity and it includes bonds that the funds also trade. In the spirit of Pavlova and Sikorskaya (2022), I assume that the compensation of the fund manager has three components.

$$W = R^x_{t+1} + b(R^x_{t+1} - R^b_{t+1}) + c = (a+b)R^x_{t+1} - bR^b_{t+1} + c$$

First, the manager gets a fraction of the return he generates on the portfolio. Second, the manager gets paid depending on the fund's performance relative to a benchmark R^b_{t+1}. In Vayanos and Vila (2021), the benchmark can be assumed to be a government bond index. Here, I assume that the benchmark is a corporate bond with maturity τ. Given the assumption of a continuum of firms subject to idiosyncratic defaults, the benchmark can be roughly thought as a general bond index for a given rating category and maturity. Third, there is a fixed fee c. The return on the manager's portfolio is

$$R^x_{t+1} = \left(W_t - z_t^{(\tau)}\right) r_t dt + z_t^{(\tau)} \left(\frac{dP^{(\tau)}_{t+1}}{P^{(\tau)}_t} - \lambda_t dt\right) + \Delta_t z_t^{(\tau)} dt$$

where, Δ_t captures the alpha that managers can generate net of a private costs of monitoring (Kashyap, Kovrijnykh, Li, & Pavlova, 2023). When habitat investors have CARA utility over next period compensation $-e^{-a^H W}$, the problem is equivalent to the standard mean-variance formulation. Substituting the compensation function, and assuming that the return on the benchmark is linear in the return on the defaultable bond, i.e. $R^b_{t+1} = \omega_b R^{(\tau)}_{t+1}$, I obtain

$$\max_{x_t^{(\tau)}} (a+b)\left[\left(W_t - x_t^{(\tau)}\right) r_t + x_t^{(\tau)} \mu_t^{(\tau)} + \Delta_t x_t^{(\tau)}\right] - b\mu_t^b + c$$
$$- \frac{a^H}{2}\left\{ (a+b)^2 (x_t^{(\tau)})^2 (\sigma_t^{(\tau)})^2 + b\left(\sigma_t^b\right)^2 - 2(a+b)b x_t^{(\tau)} \mathbb{C}\text{ov}_t\left(R^{(\tau)}_{t+1}, \omega_b R^{(\tau)}_{t+1}\right)\right\}$$

The fund manager's demand for $z_t^{(\tau)}$ is

$$z_t^{(\tau)} = \frac{1}{a+b} \cdot \frac{\mu_t^{(\tau)} - r_t - \lambda_t}{a^H \sigma_t^{2,(\tau)}} + \frac{\Delta_t}{a^H (a+b) \sigma_t^{2,(\tau)}} + \frac{b}{a+b} \cdot \omega_b$$

The third term is just a constant and does not depend neither on prices nor on risk aversion and the return variance. The quantities $\mu_t^{(\tau)}$ and $\sigma_t^{2,(\tau)}$ are determined in equilibrium. Habitat demand is an affine function of expected returns, demand shocks, here captured by the managers' skill Δ_t, and a demand intercept $\frac{b}{a+b} \cdot \omega_b$. The demand function generalizes to multiple assets.

When the risk factors follow square-root dynamics, the sensitivity to expected return takes the form

$$\frac{1}{(a+b) a^H \sigma_t^{2,(\tau)}} = \frac{1}{(a+b) a^H \lambda_t [\sigma_r^2 A_r(\tau)^2 + \sigma_\lambda^2 A_\lambda(\tau)]} = \frac{\alpha(\tau)}{\lambda_t}$$

which is the same expression as Equation (47) but with effective risk aversion $(a+b)a^H$.

A.3.2 Habitat Investors: Pension Funds & Insurance Companies

I interpret habitat investors as pension funds and insurance companies. I assume that each of these agents (i) seek to maximize expected returns but try to match the duration of their liabilities and (ii) are subject to time-varying regulatory costs that are proportional to the quantity of risky assets. There are two key ingredients. An **exogenous** liability $L^{(\tau)}$ evolves as $dL_t = L^{(\tau)} \frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$, where $\frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$ is the return on the bond with maturity τ. I interpret the liability as a "portfolio" of bonds with the same returns as the Treasury bonds in the market. Time-varying regulatory costs, linear in the positions they held in the risky asset $\psi_t X_t^{(\tau)}$. I assume that P&Is seek to maximize the objective

$$\max_{x_t^{(\tau)}} \mathbb{E}[dA_t] - \frac{a^H}{2} \mathbb{V}\text{ar}_t \left[dA_t - dL_t \right] - \psi_t x_t^{(\tau)}$$

where assets A_t evolve as

$$dA_t = \left(A_t - x_t^{(\tau)} \right) r_t dt + x_t^{(\tau)} \frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$$

Let $\mu_t^{(\tau)}$ and $\sigma_t^{(\tau)}$ denote the expected return and the volatiltiy of the bond at maturity τ. The objective can be written as

$$\max_{x_t^{(\tau)}} \left(A_t - x_t^{(\tau)} \right) r_t dt + x_t^{(\tau)} \mu_t dt - \frac{a^H}{2} \left(x_t^{(\tau)} - L^{(\tau)} \right)^2 \left(\sigma_t^{(\tau)} \right)^2 - \psi_t x_t^{(\tau)}$$

The first-order condition with respect to $x_t^{(\tau)}$ is

$$\mu_t^{(\tau)} - r_t - \psi_t - a^H \left(\sigma_t^{(\tau)} \right)^2 \left(x_t^{(\tau)} - L^{(\tau)} \right) = 0$$

Solving for $x_t^{(\tau)}$ gives

$$x_t^{(\tau)} = \frac{\mu_t^{(\tau)} - r_t}{a^H \sigma_t^{2,(\tau)}} - \frac{\psi_t}{a^H \sigma_t^{2,(\tau)}} + L^{(\tau)}$$

where $\sigma_t^{2,(\tau)}$ is the same as in the previous section.

B Solution Algorithm

Consider the system of coupled Riccati equations with constant coefficients described in Proposition 8

$$
\begin{aligned}
A'_{gr}(\tau) &= 1 - A_{gr}(\tau)\kappa^*_r - A_{g\lambda}(\tau)\kappa^*_{r\lambda} \\
A'_{cr}(\tau) &= 1 - A_{cr}(\tau)\kappa^*_r - A_{c\lambda}(\tau)\kappa^*_{r\lambda} \\
A'_{g\lambda}(\tau) &= -A_{gr}(\tau)\kappa^*_{\lambda r} - A_{g\lambda}(\tau)\kappa^*_\lambda - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 \\
A'_{c\lambda}(\tau) &= 1 - A_{c\lambda}(\tau)\kappa^*_\lambda - A_{cr}(\tau)\kappa^*_{\lambda r} - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa^*_r &\doteq \kappa_r + a\sigma_r^2 \sum_j \int_0^\infty \Psi_{jr}(\tau)A_{jr}(\tau)d\tau \\
\kappa^*_\lambda &\doteq \kappa_\lambda + a\sigma_\lambda^2 \sum_j \int_0^\infty \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] A_{j\lambda}(\tau)d\tau \\
\kappa^*_{r\lambda} &\doteq a\sigma_\lambda^2 \sum_j \int_0^\infty \Psi_{jr}(\tau)A_{j\lambda}(\tau)d\tau \\
\kappa^*_{\lambda r} &\doteq a\sigma_r^2 \sum_j \int_0^\infty \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] A_{jr}(\tau)d\tau
\end{aligned}
$$

I solve the system with the initial conditions $A_{gr}(0) = 0$, $A_{g\lambda}(0) = 0$, $A_{cr}(0) = 0$, and $A_{c\lambda}(0) = 0$. Let $\mathcal{C} \doteq [\kappa^*_r, \kappa^*_{r\lambda}, \kappa^*_{\lambda r}, \kappa^*_\lambda]^T$ denote the vector of all coefficients. I propose the following numerical procedure to solve for $\mathcal{C}$. The procedure is initialized for an initial guess $\mathcal{C}^{(0)}$.

- **Step 1**: Start with an initial guess $\mathcal{C}^{(k)}$.
- **Step 2**: Solve the coupled Riccati equations, where the coefficients $\kappa_r^{(k)}, \kappa_{r\lambda}^{(k)}, \kappa_{r\lambda}^{(k)}, \kappa_\lambda^{(k)}$ are given by the initial guess. I use a pre-programmed routine to solve the system of first-order nonlinear ordinary differential equations. Let $A_{jr}^{(k)}(\tau)$ and $A_{j\lambda}^{(k)}(\tau)$ denote the solution to the k-step system. I impose the boundary condition $A_{jr}^{(k)}(\tau) = 0$ and $A_{j\lambda}^{(k)}(\tau) = 0$.
- **Step 3**: Compute the implied coefficients $\kappa^*_r, \kappa^*_{r\lambda}, \kappa^*_{\lambda r}, \kappa^*_\lambda$, and $c^*_{\lambda\lambda}$ by plugging in $A_{jr}^{(k)}(\tau)$ and $A_{j\lambda}^{(k)}(\tau)$. This produces an updated vector of coefficients $\mathcal{C}^{(k+1)}$.
- **Step 4**: If $\|\mathcal{C}^{(k)} - \mathcal{C}^{(k+1)}\| < \varepsilon$, then terminate. Otherwise, set $k = k + 1$ and go to step 1.

C Mathematical Results

C.1 Auxiliary Lemmata and Corollaries

Lemma 1. *(Solution of System of Linear ODEs) Consider the system of linear first-order differential equations*

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$$

where A and $\boldsymbol{b}$ are constants. Suppose that A has distinct real eigenvalues and that $\boldsymbol{x}(0) = \mathbf{0}$. Let v_i denote an eigenvalue and u_i denote the corresponding eigenvector. Then

$$\boldsymbol{x} = \mathsf{u}_1\xi_1\left(\frac{e^{\mathsf{v}_1 x} - 1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n\xi_n\left(\frac{e^{\mathsf{v}_n x} - 1}{\mathsf{v}_n}\right)$$

where $\xi = P^{-1}\boldsymbol{b}$.

Proof. Diagonalize A such that

$$A = PDP^{-1} \quad : \quad P \doteq \begin{bmatrix}\mathsf{u}_1 & \mathsf{u}_2 & \ldots & \mathsf{u}_n\end{bmatrix}$$

and consider $\boldsymbol{y} = P^{-1}\boldsymbol{x}$ (with the inverse $P\boldsymbol{y} = \boldsymbol{x}$). Then,

$$\begin{aligned}\boldsymbol{y}' &= P^{-1}\boldsymbol{x}' \\ &= P^{-1}\left(A\boldsymbol{x} + \boldsymbol{b}\right) \\ &= P^{-1}AP\boldsymbol{y} + P^{-1}\boldsymbol{b} \\ &= D\boldsymbol{y} + P^{-1}\boldsymbol{b}\end{aligned}$$

Let $\xi = P^{-1}b$, and denote ξ_i the ith element of the vector ξ. It follows that

$$y_i' = \mathsf{v}_i y + \xi_i$$

Then

$$\frac{dy_i}{dx} = \mathsf{v}_i y + \xi_i \Longrightarrow dy_i = (\mathsf{v}_i y + \xi_i)dx$$

or

$$\begin{aligned}\int \frac{1}{\mathsf{v}_i y + \xi_i} dy_i = \int dx &\Longrightarrow \frac{1}{\mathsf{v}_i}\ln(\mathsf{v}_i y + \xi_i) = x + c_i \\ &\Longrightarrow \ln(\mathsf{v}_i y + \xi_i) = \mathsf{v}_i x + \mathsf{v}_i c_i \\ &\Longrightarrow y_i = \frac{e^{\mathsf{v}_i x + \mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i}\end{aligned}$$

Therefore

$$\boldsymbol{x} = P\boldsymbol{y} = \mathsf{u}_1\left(\frac{e^{\mathsf{v}_1 x + \mathsf{v}_1 c_1}}{\mathsf{v}_1} - \frac{\xi_1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n\left(\frac{e^{\mathsf{v}_i x + \mathsf{v}_n c_n}}{\mathsf{v}_n} - \frac{\xi_n}{\mathsf{v}_n}\right)$$

Solving the system with the initial condition $x_1(0) = \cdots = x_n(0) = 0$ implies

$$y_i(0) = \frac{e^{\mathsf{v}_i \cdot 0 + \mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = 0$$

or

$$e^{\mathsf{v}_i c_i} = \xi_i \Longrightarrow c_i = \frac{1}{\mathsf{v}_i}\ln \xi_i$$

Then

$$\frac{e^{\mathsf{v}_i x + \mathsf{v}_i c_1}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i x + \mathsf{v}_i \frac{1}{\mathsf{v}_i}\ln \xi_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i x} e^{\ln \xi_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{\xi_i e^{\mathsf{v}_i x}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \xi_i\left(\frac{e^{\mathsf{v}_i x} - 1}{\mathsf{v}_i}\right)$$

Hence

$$\boldsymbol{x} = \mathsf{u}_1 \xi_1\left(\frac{e^{\mathsf{v}_1 x} - 1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n \xi_n\left(\frac{e^{\mathsf{v}_n x} - 1}{\mathsf{v}_n}\right)$$

which is the desired result. ■

Lemma 2 (Expectation of Multivariate Ornstein-Uhlenbeck)**.** *Let s_τ be the state vector at time τ. Suppose that*

$$ds_t = -M^T\left(s_t - \overline{s}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}}$$

Under the risk-neutral measure $\mathbb{Q}$, q_τ is given by

$$s_\tau = e^{-M^T\tau}s_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{s}^{\mathbb{Q}} + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}}$$

where e^A is the matrix exponential operator. Further, since $B_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$

$$\mathbb{E}_0^{\mathbb{Q}}[s_\tau] = e^{-M^T\tau}s_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{s}^{\mathbb{Q}}$$

Proof. Define the demeaned process $\widetilde{q}_t = q_t - \overline{q}^{\mathbb{Q}}$. Because $\overline{q}^{\mathbb{Q}}$ is constant over time

$$d\widetilde{q}_t = dq_t \Longrightarrow d\widetilde{q}_t = -M^T\widetilde{q}_t + \Sigma dB_u^{\mathbb{Q}}$$

Standard arguments for the Ornstein-Uhlenbeck process (see e.g. Oksendal (1992)) give

$$\widetilde{q}_t = e^{-M^T\tau}\widetilde{q}_0 + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}}$$

Hence

$$
\begin{aligned}
q_\tau &= \overline{q}^{\mathbb{Q}} + e^{-M^T\tau}\left(q_0 - \overline{q}^{\mathbb{Q}}\right) + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}} \\
&= e^{-M^T\tau}q_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{q}^{\mathbb{Q}} + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}}
\end{aligned}
$$

which gives the first result. Taking expectation under $\mathbb{Q}$ gives

$$
\mathbb{E}_0^{\mathbb{Q}}[q_\tau] = e^{-M^T\tau}q_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{q}^{\mathbb{Q}}
$$

which gives the second result and completes the proof. ■

Lemma 3 (Useful Linear Operator)**.** *Let A be a (2×2) diagonal matrix and let $\boldsymbol{b}^j$ be a (2×1) column vector associated to asset class $j \in \{g, c\}$. Define the matrix function*

$$
f(A, \boldsymbol{b}^j) = \mathbb{1}^T\left(P^T\right)^{-1}AP^Tb^j
$$

Then

$$
\mathbb{1}^T\left(P^T\right)^{-1}AP^Tb^j = b_1^j\left[a_1\psi_{22}^j + a_2\psi_{12}^j\right] + b_2\left[a_1\psi_{21}^j + a_2\psi_{22}^j\right]
$$

where ψ_{11}^j, ψ_{12}^j, ψ_{21}^j and ψ_{22}^j are defined in Proposition 1.

Proof. Let

$$
P = \begin{bmatrix} \mathsf{u}_{11} & \mathsf{u}_{12} \\ \mathsf{u}_{21} & \mathsf{u}_{22} \end{bmatrix} \quad : \quad A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad : \quad \boldsymbol{b}^j = \begin{bmatrix} b_1^j \\ b_2^j \end{bmatrix}
$$

Then

$$
\begin{aligned}
\left(P^T\right)^{-1}AP^T &= \frac{1}{\det(P)}\begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{21} \\ -\mathsf{u}_{12} & \mathsf{u}_{11} \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \mathsf{u}_{11} & \mathsf{u}_{21} \\ \mathsf{u}_{12} & \mathsf{u}_{22} \end{bmatrix} \\
&= \frac{1}{\det(P)}\begin{bmatrix} \mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2 & \mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2 \\ -\mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2 & -\mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2 \end{bmatrix}
\end{aligned}
$$

It follows that

$$
\left(P^T\right)^{-1}AP^T\boldsymbol{b}^j = \frac{1}{\det(P)}\begin{bmatrix} b_1^j\left(\mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2\right) + b_2^j\left(\mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2\right) \\ b_1^j\left(-\mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2\right) + b_2^j\left(-\mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2\right) \end{bmatrix}
$$

Pre-multiplication by $\mathbb{1}^T$ yields

$$\begin{aligned}
\mathbb{1}^T \left(P^T\right)^{-1} AP^T \boldsymbol{b}^j &= \frac{b_1^j}{\det(P)} \left[\mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2 - \mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2\right] \\
&\quad + \frac{b_2^j}{\det(P)} \left[\mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2 - \mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2\right] \\
&= b_1^j \left[a_1 \frac{u_{11}(u_{22} - \mathsf{u}_{12})}{\det(P)} + a_2 \frac{\mathsf{u}_{12}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\det(P)}\right] \\
&\quad + b_2^j \left[a_1 \frac{\mathsf{u}_{21}(\mathsf{u}_{22} - \mathsf{u}_{12})}{\det(P)} + a_2 \frac{\mathsf{u}_{22}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\det(P)}\right] \\
&= b_1^j \left[a_1 \psi_{11}^j + a_2 \psi_{12}^j\right] + b_2 \left[a_1 \psi_{21}^j + a_2 \psi_{22}^j\right]
\end{aligned}$$

as desired. ■

C.2 Proofs

C.2.1 Proof of Proposition 1

Proof. Matching coefficients on the state variables s_t produces the set of first-order conditions

$$\begin{aligned}
A_g'(\tau) + MA_g(\tau) - e_1 &= 0 \\
A_c'(\tau) + MA_c(\tau) - e_1 - e_2 &= 0
\end{aligned}$$

where the $(K+2) \times (K+2)$ matrix M is the same in both systems and it is given by

$$M \doteq \Gamma^T - a \sum_j \int_0^\infty \Theta_j^T(\tau) A_j(\tau)^T - \alpha^j(\tau) A_j(\tau) A_j(\tau)^T d\tau \Sigma\Sigma^T$$

and where e_1 and e_2 are the $(K+2)$-dimensional basis vectors. I solve the two systems separately to obtain solutions for $A_g(\tau)$ and $A_c(\tau)$ taking M as a constant. Given the initial conditions $A_g(0) = A_c(0) = 0$, I first specialize Lemma (1) such that $A = -M$ and $\boldsymbol{b} = e_1$. It follows that $A_g(\tau)$ is

$$A_g(\tau) = \mathsf{u}_1 \xi_1^g \left(\frac{1 - e^{-\tau \mathsf{v}_1}}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n \xi_{K+2}^g \left(\frac{1 - e^{-\tau \mathsf{v}_{K+2}}}{\mathsf{v}_{K+2}}\right)$$

Further, let $\boldsymbol{b}^g = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$ so that $\xi^g = P^{-1}\boldsymbol{b}^g$, where P contains the eigenvectors of M. To solve for $A_c(\tau)$, I repeat the same steps as before, but specialize Lemma (1) such that $A = -M$ and $\boldsymbol{b} = e_1 + e_2$. I obtain

$$A_c(\tau) = \mathsf{u}_1 \xi_1^c \left(\frac{1 - e^{-\tau \mathsf{v}_1}}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_{K+2} \xi_{K+2}^c \left(\frac{1 - e^{-\tau \mathsf{v}_{K+2}}}{\mathsf{v}_{K+2}}\right)$$

where $\xi^c = P^{-1}\boldsymbol{b}^c$, and P contains the eigenvectors of M. Substituting $\psi_k^j \doteq \mathsf{u}_k \xi_k^j$ for $k = 1, 2, \ldots, K+2$ gives the desired result and completes the proof. ■

C.2.2 Equivalence with Affine Term Structure Models

I provide a complete proof for defaultable bonds only. The proof for Treasury bonds is identical except for the default indicator in the risk-neutral expectation. Let M be the $(n \times n)$ matrix given in equation (14). Provided that M is diagonalizable. Then

$$M = PDP^{-1}$$

where $D = \text{Diag}(\mathsf{v}_i)$ and $P = \begin{bmatrix} \mathsf{u}_1 & \dots & \mathsf{u}_n \end{bmatrix}$. This implies that the matrix exponential e^M is equal to

$$e^M = Pe^D P^{-1} = P \cdot \text{Diag}\left(e^{\mathsf{v}_i}\right) \cdot P^{-1}$$

Further, recall the definition of the constants ψ^c_{11}, ψ^c_{12}, ψ^c_{21}, and ψ^c_{22} from Proposition (1).

Armed with Lemmata (2) and (3), I consider the pricing of a zero-coupon defaultable bond with unitary payoff at time τ conditional on not defaulting, i.e. $\tau_D > \tau$. Let τ_D denote the default (stopping-time) and consider the indicator function $\mathbb{1}_{\{\tau^D > \tau\}}$. Then

$$P_0^{(\tau)} = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau r_u du}\mathbb{1}_{\{\tau^D > \tau\}}\right] = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u + \lambda_u) du}\right] \tag{1}$$

The proof consists in showing that the price of the defaultable bond given by (1) is the same as $P_0^{(\tau)} = e^{-[A(\tau)q_0 + C(\tau)]}$. To this purpose, I conjecture that, under the risk-neutral measure $\mathbb{Q}$, the state vector q_t evolves as

$$dq_t = -M^T\left(q_t - \overline{q}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}}$$

where M solves the ODE system, $\overline{q}^{\mathbb{Q}}$ is the long-term average under $\mathbb{Q}$, and $M^T\overline{q}^{\mathbb{Q}} = \chi$.

Proof. Write $\mathbb{1}^T s_t = r_t + \lambda_t$. Lemma (2) implies that, conditional on information at time 0, s_τ is multivariate Gaussian. Hence

$$\begin{aligned} P_0^{(\tau)} &= \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u+\lambda_u)du}\right] = e^{-\mathbb{E}_0^{\mathbb{Q}}[-\int_0^\tau (r_u+\lambda_u)du] + \frac{1}{2}\mathbb{V}\text{ar}_t^{\mathbb{Q}}\left(\int_0^\tau (r_u+\lambda_u)du\right)} \\ &= e^{-\mathbb{E}_0^{\mathbb{Q}}[\int_0^\tau \mathbb{1}^T s_u du] + \frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right)} \end{aligned}$$

Interchanging the expectation with the integral, the first term in the exponent can be rewritten as

$$\mathbb{E}_0^{\mathbb{Q}}\left[-\int_0^\tau \mathbb{1}^T s_u du\right] = -\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}^T s_u\right] du$$

Using Lemma (2),

$$\mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u] = \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T\left(\mathbb{I} - e^{-M^T u}\right)\overline{s}^{\mathbb{Q}} = \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T e^{-M^T u}\overline{s}^{\mathbb{Q}}$$

Using the fact that $M = PDP^{-1}$ and $M^T \overline{s}^{\mathbb{Q}} = \chi$, it follows that

$$\overline{s}^{\mathbb{Q}} = \left(M^T\right)^{-1}\chi = \left((P^{-1})^T D P^T\right)^{-1}\chi = \left(P^T\right)^{-1} D^{-1} P^T \chi$$

and, since $e^{-M^T\tau} = \left(P^{-1}\right)^T e^{-\tau D} P^T$,

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u] &= \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T e^{-M^T u}\overline{s}^{\mathbb{Q}} \\
&= \mathbb{1}^T \left(P^{-1}\right)^T e^{-\tau D} P^T s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T \left(P^{-1}\right)^T e^{-\tau D} P^T \left(P^T\right)^{-1} D^{-1} P^T \chi \\
&= \mathbb{1}^T \left(P^{-1}\right)^T e^{-\tau D} P^T s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T \left(P^{-1}\right)^T e^{-\tau D} D^{-1} P^T \chi \\
&\stackrel{\dagger}{=} r_0\left[\psi_{11}^c e^{-u\mathsf{v}_1} + \psi_{12}^c e^{-u\mathsf{v}_2}\right] + \lambda_0\left[\psi_{21}^c e^{-u\mathsf{v}_1} + \psi_{22}^c e^{-u\mathsf{v}_2}\right] \\
&\qquad + \chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2}\right] \\
&\qquad - \chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1} e^{-u\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2} e^{-u\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1} e^{-u\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2} e^{-u\mathsf{v}_2}\right]
\end{aligned}$$

where the equality † follows from repeated application of Lemma (3) after noting that $e^{-\tau D}$ and $e^{-\tau D}D^{-1}$ are diagonal matrices. Integrating with respect to time gives

$$\begin{aligned}
\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u]du &= r_0\left[\psi_{11}^c \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{12}^c \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] + \lambda_0\left[\psi_{21}^c \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{22}^c \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] \\
&\quad + \tau\left\{\chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2}\right]\right\} \\
&\quad - \chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] \\
&= r_0 A_r(\tau) + \lambda_0 A_\lambda(\tau) + \tau\left\{\chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2}\right]\right\} \\
&\quad - \chi_r\left[\frac{\psi_{11}^c}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{12}^c}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{21}^c}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{22}^c}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right]
\end{aligned}$$

However, from Proposition (1)

$$C(\tau) = \left(\int_0^\tau A(u)^T du\right)\chi - \frac{1}{2}\int_0^\tau A(u)^T \Sigma\Sigma^T A(u) du$$

Expanding the first term gives

$$\begin{aligned}
\left(\int_0^\tau A(u)du\right) &= \begin{bmatrix} \int_0^\tau \psi_{11}^c \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{12}^c \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} du \\ \int_0^\tau \psi_{21}^c \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{22}^c \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} du \end{bmatrix} \\
&= \begin{bmatrix} \frac{\psi_{11}^c}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\psi_{12}^c}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \\ \frac{\psi_{21}^c}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\psi_{22}^c}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \end{bmatrix}
\end{aligned}$$

Hence,

$$\left(\int_0^\tau A^T(u)du\right)\begin{bmatrix}\chi_r\\ \chi_\lambda\end{bmatrix} = \chi_r\left\{\frac{\psi_{11}^c}{\mathsf{v}_1}\left(\tau+\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right)+\frac{\psi_{12}^c}{\mathsf{v}_1}\left(\tau+\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)\right\}$$
$$+\chi_\lambda\left\{\frac{\psi_{21}^c}{\mathsf{v}_1}\left(\tau+\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right)+\frac{\psi_{22}^c}{\mathsf{v}_1}\left(\tau+\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)\right\}$$

By comparing this term with the expression for $\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T q_u]du$,

$$\begin{aligned}
-\int_0^\tau \mathbb{1}^T\mathbb{E}_0^{\mathbb{Q}}[q_u] &= -r_0A_r(\tau)-\lambda_0A_\lambda(\tau)-\tau\left\{\chi_r\left[\frac{\psi_{11}^c}{\nu_1}+\frac{\psi_{12}^c}{\nu_2}\right]+\chi_\lambda\left[\frac{\psi_{21}^c}{\nu_1}+\frac{\psi_{22}^c}{\nu_2}\right]\right\}\\
&\quad+\chi_r\left[\frac{\psi_{11}^c}{\nu_1}\frac{1-e^{-\nu_1\tau}}{\nu_1}+\frac{\psi_{12}^c}{\nu_2}\frac{1-e^{-\nu_2\tau}}{\nu_2}\right]+\chi_\lambda\left[\frac{\psi_{21}^c}{\nu_1}\frac{1-e^{-\nu_1\tau}}{\nu_1}+\frac{\psi_{22}^c}{\nu_2}\frac{1-e^{-\nu_2\tau}}{\nu_2}\right]\\
&= -A^T(\tau)q_0-\left(\int_0^\tau A^T(u)du\right)\chi
\end{aligned}$$

matching the linear terms in the exponent of $e^{-[A_r(\tau)r_t+A_\lambda(\tau)\lambda_t+C(\tau)]}$. This gives the desired result and completes the first part of the proof. The second part of the proof matches the variance term. To compute is

$$\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T q_u du\right)=\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T\left(\int_0^u e^{-M^T(u-v)}\Sigma dB_v^{\mathbb{Q}}\right)du\right)$$

Note that

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left[\int_0^\tau \mathbb{1}^T q_u du\right] &= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(\mathbb{1}^T q_u,\mathbb{1}^T q_{u'})dudu'\\
&= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(r_u+\lambda_u,r_{u'}+\lambda_{u'})dudu'\\
&= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(r_u,r_{u'})dudu'+\int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(\lambda_u,\lambda_{u'})dudu'
\end{aligned}$$

The covariance functions of each of the components of q_u separately. Recall that

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(r_u,r_{u'}) &= \mathbb{C}\mathrm{ov}\left\{e_1^T\left(\int_0^u e^{-M^T(u-v)}\Sigma dB_v^{\mathbb{Q}}\right),e_1^T\left(\int_0^{u'} e^{-M^T(u'-v)}\Sigma dB_v^{\mathbb{Q}}\right)\right\}\\
&= \mathbb{C}\mathrm{ov}\left\{e_1^T\left(\int_0^{\min\{u,u'\}} e^{-M^T(u-v)}\Sigma dB_v^{\mathbb{Q}}\right),e_1^T\left(\int_0^{\min\{u,u'\}} e^{-M^T(u'-v)}\Sigma dB_v^{\mathbb{Q}}\right)\right\}\\
&= \mathbb{C}\mathrm{ov}\left(\int_0^{\min\{u,u'\}} e_1^Te^{-M^T(u-v)}\Sigma dB_v^{\mathbb{Q}},\int_0^{\min\{u,u'\}} e_1^Te^{-M^T(u'-v)}\Sigma dB_v^{\mathbb{Q}}\right)\\
&= \sigma_r^2(\psi_{11}^c)^2e^{-\mathsf{v}_1(u+u')}\int_0^{\min\{u,u'\}}e^{2\mathsf{v}_1v}dv+\sigma_r^2\psi_{11}^c\psi_{12}^ce^{-\mathsf{v}_1u-\mathsf{v}_2u'}\int_0^{\min\{u,u'\}}e^{(\mathsf{v}_1+\mathsf{v}_2)v}dv\\
&\quad+\sigma_r^2\psi_{12}^c\psi_{11}^ce^{-\mathsf{v}_2u-\mathsf{v}_1u'}\int_0^{\min\{u,u'\}}e^{(\mathsf{v}_1+\mathsf{v}_2)v}dv+\sigma_r^2(\psi_{11}^c)^2e^{-\mathsf{v}_2(u+u')}\int_0^{\min\{u,u'\}}e^{2\mathsf{v}_2v}dv
\end{aligned}$$

whereas the second line uses Lemma (3) in combination with the Itô Isometry. Each of the integrals is a Riemann integral of the form

$$\int_0^{\min\{u,u'\}} e^{\mathsf{v}v}dv = \frac{1}{\mathsf{v}}e^{\mathsf{v}v}\Big|_0^{\min\{u,u'\}} = \frac{1}{\mathsf{v}}\left(e^{\mathsf{v}\min\{u,u'\}} - 1\right)$$

Hence

$$\begin{aligned}\mathbb{C}\mathrm{ov}(r_u, r_{u'}) = \sigma_r^2(\psi_{11}^c)^2 e^{-\mathsf{v}_1(u+u')}\frac{1}{2\mathsf{v}_1}\left(e^{2\mathsf{v}_1\min\{u,u'\}} - 1\right) + \sigma_r^2\psi_{11}^c\psi_{12}^c e^{-\mathsf{v}_1 u - \mathsf{v}_2 u'}\frac{e^{(\mathsf{v}_1+\mathsf{v}_2)\min\{u,u'\}} - 1}{\mathsf{v}_1+\mathsf{v}_2} \\ + \sigma_r^2\psi_{12}^c\psi_{11}^c e^{-\mathsf{v}_2 u - \mathsf{v}_1 u'}\frac{e^{(\mathsf{v}_1+\mathsf{v}_2)\min\{u,u'\}} - 1}{\mathsf{v}_1+\mathsf{v}_2} + \sigma_r^2(\psi_{11}^c)^2 e^{-\mathsf{v}_2(u+u')}\frac{1}{2\mathsf{v}_2}\left(e^{2\mathsf{v}_2\min\{u,u'\}} - 1\right)\end{aligned}$$

Therefore, for the case that $u > u'$

$$\frac{1}{2}\int_t^{t+\tau}\int_t^{t+\tau}\mathbb{C}\mathrm{ov}(r_u, r_{u'})dudu' = \frac{\sigma_r^2(\psi_{11}^c)^2}{2\mathsf{v}_1}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\}$$

Then

$$\begin{aligned}\int_0^\tau\int_0^u \mathbb{C}\mathrm{ov}(r_u, r_{u'})du'du = \frac{\sigma_r^2(\psi_{11}^c)^2}{2\mathsf{v}_1^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_r^2(\psi_{12}^c)^2}{2\mathsf{v}_2^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \\ + \left[\frac{1}{\mathsf{v}_1} + \frac{1}{\mathsf{v}_2}\right]\frac{\sigma_r^2\psi_{11}^c\psi_{12}^c}{\mathsf{v}_1+\mathsf{v}_2}\left\{\tau + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)\tau}}{(\mathsf{v}_1+\mathsf{v}_2)} - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}\end{aligned}$$

By symmetry

$$\begin{aligned}\int_0^\tau\int_0^u \mathbb{C}\mathrm{ov}(\lambda_u, \lambda_{u'})du'du = \frac{\sigma_\lambda^2(\psi_{21}^c)^2}{2\mathsf{v}_1^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_\lambda^2(\psi_{22}^c)^2}{2\mathsf{v}_2^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \\ + \left[\frac{1}{\mathsf{v}_1} + \frac{1}{\mathsf{v}_2}\right]\frac{\sigma_r^2\psi_{21}^c\psi_{22}^c}{\mathsf{v}_1+\mathsf{v}_2}\left\{\tau + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)\tau}}{(\mathsf{v}_1+\mathsf{v}_2)} - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}\end{aligned}$$

As a result, these two expressions give

$$\frac{1}{2}\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right) = \int_0^\tau\int_0^u \mathbb{C}\mathrm{ov}(r_u, r_{u'})du'du + \int_0^\tau\int_0^u \mathbb{C}\mathrm{ov}(\lambda_u, \lambda_{u'})du'du$$

The final step is to show that the second element of $C_c(\tau)$ is equal to these two terms. Note that

$$\frac{1}{2}\int_0^\tau A_c(u)^T\Sigma\Sigma^T A_c(u)du = \frac{1}{2}\int_0^\tau \sigma_r^2 A_{cr}^2(u)du + \frac{1}{2}\int_0^\tau \sigma_\lambda^2 A_{c\lambda}^2(u)du$$

Then, using the expressions for $A_r(\tau)$ and $A_\lambda(\tau)$

$$\frac{1}{2}\sigma_r^2\int_0^\tau A_{cr}^2(u)du = \frac{\sigma_r^2(\psi_{11}^c)^2}{2\mathsf{v}_1{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_r^2(\psi_{12}^c)^2}{2\mathsf{v}_2{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}$$
$$+ \frac{\psi_{11}^c\psi_{12}^c}{\mathsf{v}_1\mathsf{v}_2}\left\{\tau - \frac{1-e^{-\mathsf{v}_1 u}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2 u}}{\mathsf{v}_2} + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)}}{\mathsf{v}_1+\mathsf{v}_2}\right\}$$
$$\frac{1}{2}\sigma_\lambda^2\int_0^\tau A_{c\lambda}^2(u)du = \frac{\sigma_\lambda^2(\psi_{21}^c)^2}{2\mathsf{v}_1{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_r^2(\psi_{22}^c)^2}{2\mathsf{v}_2{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}$$
$$+ \frac{\psi_{22}^c\psi_{21}^c}{\mathsf{v}_1\mathsf{v}_2}\left\{\tau - \frac{1-e^{-\mathsf{v}_1 u}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2 u}}{\mathsf{v}_2} + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)}}{\mathsf{v}_1+\mathsf{v}_2}\right\}$$

which clearly match the expressions given above. As a result

$$\frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right) = \frac{1}{2}\int_0^\tau A_c(u)^T\Sigma\Sigma^T A_c(u)du$$

from which it follows that

$$\begin{aligned}P_{c,0}^{(\tau)} = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u+\lambda_u)du}\right] &= e^{-\mathbb{E}_0^{\mathbb{Q}}[\int_0^\tau \mathbb{1}^T s_u du]+\frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}(\int_0^\tau \mathbb{1}^T s_u du)}\\ &= e^{-A_c^T(\tau)s_0-(\int_0^\tau A_c^T(u)du)\chi+\frac{1}{2}\int_0^\tau A_c(u)^T\Sigma\Sigma^T A_c(u)du}\\ &= e^{-[A_{cr}(\tau)r_0+A_{c\lambda}(\tau)\lambda_0+C_c(\tau)]}\end{aligned}$$

This is the desired result and it concludes the proof. ∎

C.2.3 Proof of Proposition 2

Proof. I specialize Proposition (1) to $K=0$ and to independent risk factors. Suppose that Γ and Σ are diagonal and that the short rate and the default intensity are the only risk factors. Define

$$\kappa_r^* = \kappa_r - a\sigma_r^2\int_0^\infty\sum_j\left(\rho^j(\tau) - \alpha^j(\tau)A_{jr}(\tau) + \gamma^j(\tau)A_{-jr}(\tau)\right)A_{jr}(\tau)d\tau$$
$$\kappa_{\lambda r} = a\sigma_\lambda^2\int_0^\infty\sum_j\left(\rho^j(\tau) - \alpha^j(\tau)A_{jr}(\tau) + \gamma^j(\tau)A_{-jr}(\tau)\right)A_{j\lambda}(\tau)d\tau$$
$$\kappa_{r\lambda} = a\sigma_r^2\int_0^\infty\sum_j\left(\phi^j(\tau) - \alpha^j(\tau)A_{j\lambda}(\tau) + \gamma^j(\tau)A_{-j\lambda}(\tau)\right)A_{jr}(\tau)d\tau$$
$$\kappa_\lambda^* = \kappa_\lambda - a\sigma_\lambda^2\int_0^\infty\sum_j\left(\phi^j(\tau) - \alpha^j(\tau)A_{j\lambda}(\tau) + \gamma^j(\tau)A_{-j\lambda}(\tau)\right)A_{j\lambda}(\tau)d\tau$$

so that the matrix M can be written as

$$M = \begin{bmatrix}\kappa_r^* & -\kappa_{\lambda r}\\ -\kappa_{r\lambda} & \kappa_\lambda^*\end{bmatrix}$$

The notation emphasizes that the martingale dynamics of s_t depend on its transpose M^T. From Proposition (1), the solution to the system is

$$A_g(\tau) = \psi_1^g \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_2^g \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}$$
$$A_c(\tau) = \psi_1^c \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_2^c \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}$$

where $\psi_k^j = \mathsf{u}_k \xi_k^j$ and $\xi^j = P^{-1} b^j$. Let the matrix containing the right eigenvectors of M be

$$P = \begin{bmatrix} \mathsf{u}_1 & \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{u}_{11} & \mathsf{u}_{12} \\ \mathsf{u}_{21} & \mathsf{u}_{22} \end{bmatrix}$$

The eigenvalues of M solve

$$\det(M - \mathsf{v}\mathbb{I}) = \det \begin{bmatrix} \kappa_r^* - \mathsf{v} & -\kappa_{\lambda r} \\ -\kappa_{r\lambda} & \kappa_\lambda^* - \mathsf{v} \end{bmatrix} = 0$$

The characteristic equation is

$$(\kappa_r^* - \mathsf{v})(\kappa_\lambda^* - \mathsf{v}) + \kappa_{\lambda r}\kappa_{r\lambda} = 0$$

The solutions are

$$\mathsf{v}_{1,2} = \frac{\kappa_r^* + \kappa_\lambda^* \pm \sqrt{(\kappa_r^* - \kappa_\lambda^*)^2 - 4\kappa_{\lambda r}\kappa_{r\lambda}}}{2}$$

where v_1 is the largest root, and v_2 is the smallest root. Given the eigenvalues, I solve for the corresponding eigenvectors as

$$M\mathsf{u}_1 = \mathsf{v}_1\mathsf{u}_1 \iff (M - \mathsf{v}_1\mathbb{I})\mathsf{u}_1 = 0$$

This gives the system of equations

$$(\kappa_r^* - \mathsf{v}_1)\mathsf{u}_{11} - \kappa_{\lambda r}\mathsf{u}_{21} = 0$$
$$-\kappa_{r\lambda}\mathsf{u}_{11} + (\kappa_\lambda^* - \mathsf{v}_1)\mathsf{u}_{21} = 0$$

The system of equation implies $\frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_1} = \frac{\kappa_\lambda^* - \mathsf{v}_1}{\kappa_{r\lambda}}$, so that

$$\mathsf{u}_{11} = \frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_1}\mathsf{u}_{21} \Longrightarrow \mathsf{u}_1 = c_1 \begin{bmatrix} 1 \\ \frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_1} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \frac{\kappa_\lambda^* - \mathsf{v}_1}{\kappa_{r\lambda}} \end{bmatrix}$$

Repeating the same steps for the eigenvector corresponding to the second eigenvalue yields the system

$$(\kappa_r^* - \mathsf{v}_1)\mathsf{u}_{12} - \kappa_{\lambda r}\mathsf{u}_{22} = 0$$
$$-\kappa_{r\lambda}\mathsf{u}_{12} + (\kappa_\lambda^* - \mathsf{v}_1)\mathsf{u}_{22} = 0$$

The system implies $\frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_2} = \frac{\kappa_\lambda^* - \mathsf{v}_2}{\kappa_{r\lambda}}$, so that

$$\mathsf{u}_{12} = \frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_2}\mathsf{u}_{22} \Longrightarrow \mathsf{u}_2 = c_2 \begin{bmatrix} 1 \\ \frac{\kappa_{\lambda r}}{\kappa_r^* - \mathsf{v}_2} \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ \frac{\kappa_\lambda^* - \mathsf{v}_2}{\kappa_{r\lambda}} \end{bmatrix}$$

I therefore conclude that

$$P = \begin{bmatrix} \frac{\kappa_\lambda^* - \mathsf{v}_1}{\kappa_{r\lambda}} & 1 \\ 1 & \frac{\kappa_{r\lambda}}{\kappa_\lambda^* - \mathsf{v}_2} \end{bmatrix}$$

Given the matrix P and the eigenvectors $\mathsf{v}_{1,2}$, the coefficients ξ^j are

$$\xi^g = P^{-1}b^g = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{12} \\ -\mathsf{u}_{21} & \mathsf{u}_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} \\ -\mathsf{u}_{21} \end{bmatrix}$$

and

$$\xi^c = P^{-1}b^c = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{12} \\ -\mathsf{u}_{21} & \mathsf{u}_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} - \mathsf{u}_{12} \\ -\mathsf{u}_{21} + \mathsf{u}_{11} \end{bmatrix}$$

Therefore

$$\begin{aligned} A_g(\tau) &= \psi_1^g \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_2^g \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \\ &= \mathsf{u}_1 \cdot \frac{\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \mathsf{u}_2 \cdot \frac{-\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \end{aligned}$$

and

$$\begin{aligned} A_c(\tau) &= \psi_1^c \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_2^c \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \\ &= \mathsf{u}_1 \cdot \frac{\mathsf{u}_{22} - \mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \mathsf{u}_2 \cdot \frac{-\mathsf{u}_{21} + \mathsf{u}_{11}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \end{aligned}$$

which gives

$$\begin{aligned} A_{g\lambda}(\tau) &= \mathsf{u}_{21} \cdot \frac{\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_1}}{\mathsf{v}_1} + \mathsf{u}_{22} \cdot \frac{-\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \\ &= \frac{\mathsf{u}_{22}\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \right) \end{aligned}$$

Substituting in the elements of P gives

$$A_{g\lambda}(\tau) = \frac{\kappa_{r\lambda}}{\mathsf{v}_2 - \mathsf{v}_1}\left(\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)$$

which is the desired result and completes the proof. ■

C.2.4 Proof of Proposition 3

Proof. From Proposition (1), it immediately follows that

$$A'_g(\tau) = \psi_1^g e^{-\mathsf{v}_1\tau} + \psi_2^g e^{-\mathsf{v}_2\tau}$$
$$A'_c(\tau) = \psi_1^c e^{-\mathsf{v}_1\tau} + \psi_2^c e^{-\mathsf{v}_2\tau}$$

Hence

$$\begin{aligned} A'_{gr}(\tau) &= \mathsf{u}_{11}\xi_1^g e^{-\mathsf{v}_1\tau} + \mathsf{u}_{12}\xi_2^g e^{-\mathsf{v}_2\tau} \\ &= \frac{\mathsf{u}_{11}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} e^{-\mathsf{v}_1\tau} - \frac{\mathsf{u}_{21}\mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} e^{-\mathsf{v}_2\tau} \\ &= \frac{\kappa_\lambda^* - \mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1} e^{-\mathsf{v}_1\tau} - \frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1} e^{-\mathsf{v}_2\tau} \end{aligned}$$

and

$$\begin{aligned} A'_{cr}(\tau) &= \mathsf{u}_{11}\xi_1^c e^{-\mathsf{v}_1\tau} + \mathsf{u}_{12}\xi_2^c e^{-\mathsf{v}_2\tau} \\ &= \frac{\mathsf{u}_{11}(\mathsf{u}_{22} - \mathsf{u}_{12})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} e^{-\mathsf{v}_1\tau} + \frac{\mathsf{u}_{12}(-\mathsf{u}_{21} + \mathsf{u}_{11})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} e^{-\mathsf{v}_2\tau} \\ &= A'_{gr}(\tau) - \frac{\mathsf{u}_{11}\mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}}\left(e^{-\mathsf{v}_1\tau} - e^{-\mathsf{v}_2\tau}\right) \\ &= A'_{gr}(\tau) - \frac{\kappa_\lambda^* - \mathsf{v}_1}{\kappa_{r\lambda}} \frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1}\left(e^{-\mathsf{v}_1\tau} - e^{-\mathsf{v}_2\tau}\right) \end{aligned}$$

as it was to be shown. ■

C.2.5 Proof of Proposition 4

Proof. Similar arguments as for Proposition (2) give

$$\begin{aligned} A_{c\lambda}(\tau) &= A_{g\lambda}(\tau) - \frac{\mathsf{u}_{21}\mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\mathsf{u}_{22}\mathsf{u}_{11}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \\ &= A_{g\lambda}(\tau) - \frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1} \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\kappa_\lambda^* - \mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \\ &= A_{g\lambda}(\tau) - \frac{\kappa_\lambda^*}{\mathsf{v}_2 - \mathsf{v}_1}\left(\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right) + \frac{\mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1} \frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} - \frac{\mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1} \frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2} \end{aligned}$$

Subtracting $A_{g\lambda}(\tau)$ from both sides gives

$$A_{c\lambda}(\tau) - A_{g\lambda}(\tau) = -\frac{\kappa_\lambda^* - \mathsf{v}_2}{\mathsf{v}_2 - \mathsf{v}_1}\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\kappa_\lambda^* - \mathsf{v}_1}{\mathsf{v}_2 - \mathsf{v}_1}\frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}$$

If $a = 0$, $\kappa_\lambda = \kappa_\lambda^*$ and $\kappa_r = \kappa_r$, so that $\mathsf{v}_2 = \kappa_\lambda$ and $\mathsf{v}_2 = \kappa_r$. It follows that

$$A_{c\lambda}^*(\tau) - A_{g\lambda}^*(\tau) = \frac{\kappa_\lambda - \kappa_r}{\kappa_\lambda - \kappa_r}\frac{1 - e^{-\kappa_\lambda\tau}}{\kappa_\lambda} = \frac{1 - e^{-\kappa_\lambda\tau}}{\kappa_\lambda}$$

■

C.2.6 Proof of Proposition 5

Proof. The result is immediate from specializing Proposition (1) to $K = 0$ and diagonal Σ and Γ. The desired result follows from subtracting Treasury excess returns from corporate bond excess returns. ■

C.2.7 Proof of Proposition 6

Proof. Instantaneous expected returns on Treasury and corporate bonds follows

$$\frac{dP_{j,t}^{(\tau)}}{P_{j,t}^{(\tau)}} = \mu_{j,t}^{(\tau)}dt - A_{jr}(\tau)\sqrt{\lambda_t}\sigma_r dB_{r,t} - A_{j,\lambda}(\tau)\sqrt{\lambda_t}\sigma_\lambda dB_{\lambda,t}$$

It follows that

$$\mathbb{E}_t[dW_t] = \left(W_t - \int_0^\infty \sum_j x_{j,t}^{(\tau)}d\tau\right)r_t dt + \left(\int_0^\infty x_{g,t}^{(\tau)}\mu_{g,t}^{(\tau)}d\tau\right)dt + \left(\int_0^\infty x_{c,t}^{(\tau)}\left(\mu_{c,t}^{(\tau)} - \lambda_t\right)d\tau\right)dt$$

and

$$\begin{aligned}\mathbb{V}\mathrm{ar}_t[dW_t] = \sigma_r^2\lambda_t\left[\int_0^\infty x_{g,t}^{(\tau)}A_{gr}(\tau)d\tau + \int_0^\infty x_{c,t}^{(\tau)}A_{cr}(\tau)d\tau\right]^2 dt \\ + \sigma_\lambda^2\lambda_t\left[\int_0^\infty x_{g,t}^{(\tau)}A_{g\lambda}(\tau)d\tau + \int_0^\infty x_{c,t}^{(\tau)}A_{c\lambda}(\tau)d\tau\right]^2 dt\end{aligned}$$

Hence, the problem becomes

$$\begin{aligned}&\max_{\{x_{j,t}^{(\tau)}\}_{\tau\in\{0,\infty\}}^{j\in\{g,c\}}} \left(W_t - \int_0^\infty \sum_j x_{j,t}^{(\tau)}d\tau\right)r_t + \left(\int_0^\infty x_{g,t}^{(\tau)}\mu_{g,t}^{(\tau)}d\tau\right) + \left(\int_0^\infty x_{c,t}^{(\tau)}\left(\mu_{c,t}^{(\tau)} - \lambda_t\right)d\tau\right) \\ &- \frac{a}{2}\left\{\sigma_r^2\lambda_t\left[\int_0^\infty x_{g,t}^{(\tau)}A_{gr}(\tau)d\tau + \int_0^\infty x_{c,t}^{(\tau)}A_{cr}(\tau)d\tau\right]^2 + \sigma_\lambda^2\lambda_t\left[\int_0^\infty x_{g,t}^{(\tau)}A_{g\lambda}(\tau)d\tau + \int_0^\infty x_{c,t}^{(\tau)}A_{c\lambda}(\tau)d\tau\right]^2\right\}\end{aligned}$$

The first-order conditions are

$$\mu_{g,t} - r_t = a\sigma_r^2\lambda_t A_{gr}(\tau)\left[\sum_j\int_0^\infty x_{j,t}^{(\tau)}A_{jr}(\tau)d\tau\right] + a\sigma_\lambda^2\lambda_t A_{g\lambda}(\tau)\left[\sum_j\int_0^\infty x_{j,t}^{(\tau)}A_{j\lambda}(\tau)d\tau\right]$$

and

$$\mu_{c,t} - r_t = \lambda_t + a\sigma_r^2 \lambda_t A_{cr}(\tau) \left[\sum_j \int_0^\infty x_{j,t}^{(\tau)} A_{jr}(\tau) d\tau \right] + a\sigma_\lambda^2 \lambda_t A_{c\lambda}(\tau) \left[\sum_j \int_0^\infty x_{j,t}^{(\tau)} A_{j\lambda}(\tau) d\tau \right]$$

The market price of risk associated to factor $s \in \{r, \lambda\}$ is

$$\eta_{s,t} = \left[\sum_j \int_0^\infty x_{j,t}^{(\tau)} A_{js}(\tau) d\tau \right]$$

I conclude that

$$\begin{aligned}
\mu_{g,t} - r_t &= a\sigma_r^2 \lambda_t A_{gr}(\tau) \eta_{r,t} + a\sigma_\lambda^2 \lambda_t A_{g\lambda}(\tau) \eta_{\lambda,t} \\
\mu_{c,t} - r_t &= \lambda_t + a\sigma_r^2 \lambda_t A_{cr}(\tau) \eta_{r,t} + a\sigma_\lambda^2 \lambda_t A_{c\lambda}(\tau) \eta_{\lambda,t}
\end{aligned}$$

which gives the desired result.

■

C.2.8 Proof of Proposition 7

The coefficients on Treasury hedgers' demand are given by

$$\begin{aligned}
\Psi_{gr}(\tau) &\doteq \frac{A'_{gr}(\tau) + A_{gr}(\tau)\kappa_r - (1 + \theta_g(\tau))}{a^g \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2 \right]} \\
\Psi_{g\lambda}(\tau) &\doteq \frac{A'_{g\lambda}(\tau) + A_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2} A_{gr}(\tau)^2 \sigma_r^2 + \frac{1}{2} A_{g\lambda}(\tau)^2 \sigma_\lambda^2}{a^g \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2 \right]} \\
\Psi_{g0}(\tau) &\doteq \frac{C'_g(\tau) - A_{gr}(\tau)\kappa_r \overline{r} - A_{g\lambda}(\tau)\kappa_\lambda \overline{\lambda}}{a^g \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2 \right]}
\end{aligned}$$

and

$$\Omega_g(\tau) \doteq \frac{\delta_{r,g}^{(\tau)} A_{gr}(\tau)\sigma_r + \delta_{\lambda,g}^{(\tau)} A_{g\lambda}(\tau)\sigma_\lambda}{A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2}$$

The coefficients on corporate bond hedgers' demand are given by

$$\begin{aligned}
\Psi_{cr}(\tau) &\doteq \frac{A'_{cr}(\tau) + A_{cr}(\tau)\kappa_r - (1 + \theta_c(\tau))}{a^c \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2 \right]} \\
\Psi_{c\lambda}(\tau) &\doteq \frac{A'_{c\lambda}(\tau) + A_{c\lambda}(\tau)\kappa_\lambda + \frac{1}{2} A_{cr}(\tau)^2 \sigma_r^2 + \frac{1}{2} A_{c\lambda}(\tau)^2 \sigma_\lambda^2 - 1}{a^c \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2 \right]} \\
\Psi_{c0}(\tau) &\doteq \frac{C'_c(\tau) - A_{cr}(\tau)\kappa_r \overline{r} - A_{c\lambda}(\tau)\kappa_\lambda \overline{\lambda}}{a^c \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2 \right]}
\end{aligned}$$

and

$$\Omega_c(\tau) \doteq \frac{\delta_{r,c}^{(\tau)} A_{cr}(\tau)\sigma_r + \delta_{\lambda,c}^{(\tau)} A_{c\lambda}(\tau)\sigma_\lambda}{A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2}$$

Proof. I solve the problem of the corporate bond hedgers first. Corporate bond hedgers solve

$$\max_{z_{c,t}^{(\tau)}} \mathbb{E}_t\left(dW_{c,t}\right) - \frac{a^c}{2}\mathbb{V}\text{ar}_t\left(dW_{c,t}\right)$$

subject to the budget constraint where

$$dW_{c,t} = \left(W_{c,t} - z_{c,t}^{(\tau)}\right)(1+\theta_c(\tau))r_t dt + z_{c,t}^{(\tau)}\left(\frac{dP_{c,t}^{(\tau)}}{P_{c,t}^{(\tau)}} - \lambda_t dt\right) + \sqrt{\lambda_t}\cdot\delta_c(\tau)dB_t$$

Then, because dB_t is mean zero, it follows that

$$\mathbb{E}_t\left(dW_{c,t}\right) = \left(W_{c,t} - z_{c,t}^{(\tau)}\right)(1+\theta_c(\tau))r_t dt + z_{c,t}^{(\tau)}\left(\mu_{c,t}^{(\tau)} - \lambda_t\right)dt$$

and

$$\begin{aligned}\mathbb{V}\text{ar}_t\left(W_{c,t}\right) &= \mathbb{V}\text{ar}_t\left(z_{c,t}^{(\tau)}\left[-A_{jr}(\tau)\sqrt{\lambda_t}\sigma_r dB_{r,t} - A_{j,\lambda}(\tau)\sqrt{\lambda_t}\sigma_\lambda dB_{\lambda,t}\right] + \sqrt{\lambda_t}\cdot\delta_c^{(\tau)}dB_t\right)\\ &= \lambda_t\left(\delta_{cr}(\tau) - z_{c,t}^{(\tau)}A_{cr}(\tau)\sigma_r\right)^2 dt + \lambda_t\left(\delta_{c\lambda}(\tau) - z_{c,t}^{(\tau)}A_{c\lambda}(\tau)\sigma_\lambda\right)^2 dt\end{aligned}$$

Thus, the problem becomes

$$\begin{aligned}\max_{z_{c,t}^{(\tau)}} &\left(W_t^c - z_{c,t}^{(\tau)}\right)(1+\theta_c(\tau))r_t + z_{c,t}^{(\tau)}\left(\mu_{c,t}^{(\tau)} - \lambda_t\right)\\ &- \frac{a^c}{2}\left\{\lambda_t\left(\delta_{cr}(\tau) - z_{c,t}^{(\tau)}A_{cr}(\tau)\sigma_r\right)^2 + \lambda_t\left(\delta_{c\lambda}(\tau) - z_{c,t}^{(\tau)}A_{c\lambda}(\tau)\sigma_\lambda\right)^2\right\}\end{aligned}$$

The first-order condition of the corporate hedger is

$$\begin{aligned}&\mu_{c,t}^{(\tau)} - \lambda_t - (1+\theta_c(\tau))r_t =\\ &\quad a^c\lambda_t\left\{\left(\delta_{r,c}^{(\tau)} - z_{c,t}^{(\tau)}A_{cr}(\tau)\sigma_r\right)(-A_{cr}(\tau)\sigma_r) + \left(\delta_{\lambda,c}^{(\tau)} - z_{c,t}^{(\tau)}A_{c\lambda}(\tau)\sigma_\lambda\right)(-A_{c\lambda}(\tau)\sigma_\lambda)\right\}\end{aligned}$$

Solving for $z_{c,t}^{(\tau)}$ yields

$$z_{c,t}^{(\tau)} = \frac{\mu_{c,t}^{(\tau)} - \lambda_t - (1+\theta_c(\tau))r_t}{a^c\lambda_t\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]} + \frac{\delta_{r,c}^{(\tau)}A_{cr}(\tau)\sigma_r + \delta_{\lambda,c}^{(\tau)}A_{c\lambda}(\tau)\sigma_\lambda}{A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2}$$

Then, since

$$\begin{aligned}\mu_{c,t}^{(\tau)} = A_{cr}'(\tau)r_t + A_{c\lambda}'(\tau)\lambda_t + C_c'(\tau) + A_{cr}(\tau)\kappa_r(r_t - \overline{r})&\\ + A_{c\lambda}(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) + \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2\lambda_t + \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2\lambda_t&\end{aligned}$$

I conclude that $z_{c,t}^{(\tau)}$ can be written as

$$z_{c,t}^{(\tau)} = \frac{1}{\lambda_t}\left(\Psi_{cr}(\tau)r_t + \Psi_{c\lambda}(\tau)\lambda_t + \Psi_{c,0}(\tau)\right) + \Omega_c(\tau)$$

where

$$\begin{aligned}
\Psi_{cr}(\tau) &\doteq \frac{A'_{cr}(\tau) + A_{cr}(\tau)\kappa_r - (1+\theta_c(\tau))}{a^c\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]} \\
\Psi_{c\lambda}(\tau) &\doteq \frac{A'_{c\lambda}(\tau) + A_{c\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2 - 1}{a^c\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]} \\
\Psi_{c0}(\tau) &\doteq \frac{C'_c(\tau) - A_{cr}(\tau)\kappa_r\overline{r} - A_{c\lambda}(\tau)\kappa_\lambda\overline{\lambda}}{a^c\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]}
\end{aligned}$$

and

$$\Omega_c(\tau) \doteq \frac{\delta_{r,c}^{(\tau)}A_{cr}(\tau)\sigma_r + \delta_{\lambda,c}^{(\tau)}A_{c\lambda}(\tau)\sigma_\lambda}{A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2}$$

The problem of the Treasury hedgers is symmetric, except for the fact that no Treasury bond default. Treasury demand is

$$z_{g,t}^{(\tau)} = \frac{\mu_{g,t}^{(\tau)} - (1+\theta_g(\tau))r_t}{a^g\lambda_t\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]} + \frac{\delta_{r,g}^{(\tau)}A_{gr}(\tau)\sigma_r + \delta_{\lambda,g}^{(\tau)}A_{g\lambda}(\tau)\sigma_\lambda}{A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2}$$

which can also be written as

$$z_{g,t}^{(\tau)} = \frac{1}{\lambda_t}\left(\Psi_{gr}(\tau)r_t + \Psi_{g\lambda}(\tau)\lambda_t + \Psi_{g,0}(\tau)\right) + \Omega_g(\tau)$$

where

$$\begin{aligned}
\Psi_{gr}(\tau) &\doteq \frac{A'_{gr}(\tau) + A_{gr}(\tau)\kappa_r - (1+\theta_g(\tau))}{a^g\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]} \\
\Psi_{g\lambda}(\tau) &\doteq \frac{A'_{g\lambda}(\tau) + A_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2}{a^g\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]} \\
\Psi_{g0}(\tau) &\doteq \frac{C'_g(\tau) - A_{gr}(\tau)\kappa_r\overline{r} - A_{g\lambda}(\tau)\kappa_\lambda\overline{\lambda}}{a^g\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]}
\end{aligned}$$

and

$$\Omega_g(\tau) \doteq \frac{\delta_{r,g}^{(\tau)}A_{gr}(\tau)\sigma_r + \delta_{\lambda,g}^{(\tau)}A_{g\lambda}(\tau)\sigma_\lambda}{A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2}$$

■

C.2.9 Proof of Proposition 8

Proof. Recall that

$$\mu_{g,t}^{(\tau)} = A'_{gr}(\tau)r_t + A'_{g\lambda}(\tau)\lambda_t + C'_g(\tau) + A_{gr}(\tau)\kappa_r(r_t - \overline{r}) + A_{g\lambda}(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) + \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2\lambda_t + \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2\lambda_t$$

$$\mu_{c,t}^{(\tau)} = A'_{cr}(\tau)r_t + A'_{c\lambda}(\tau)\lambda_t + C'_c(\tau) + A_{cr}(\tau)\kappa_r(r_t - \overline{r}) + A_{c\lambda}(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) + \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2\lambda_t + \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2\lambda_t$$

Matching coefficients on r_t and rearranging gives

$$A'_{gr}(\tau) = 1 - A_{gr}(\tau)\left[\kappa_r + a\sigma_r^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{jr}(\tau)d\tau\right] - A_{g\lambda}(\tau)a\sigma_\lambda^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{j\lambda}(\tau)d\tau$$
$$A'_{cr}(\tau) = 1 - A_{cr}(\tau)\left[\kappa_r + a\sigma_r^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{jr}(\tau)d\tau\right] - A_{c\lambda}(\tau)a\sigma_\lambda^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{j\lambda}(\tau)d\tau$$

Matching coefficients on λ_t and rearranging gives

$$A'_{g\lambda}(\tau) = -A_{g\lambda}(\tau)\left[\kappa_\lambda + a\sigma_\lambda^2A_{g\lambda}(\tau)\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{j\lambda}(\tau)d\tau\right] - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 - a\sigma_r^2A_{gr}(\tau)\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{jr}(\tau)d\tau$$
$$A'_{c\lambda}(\tau) = 1 - A_{c\lambda}(\tau)\left[\kappa_\lambda + a\sigma_\lambda^2A_{c\lambda}(\tau)\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{j\lambda}(\tau)d\tau\right] - A_{cr}(\tau)a\sigma_r^2\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{jr}(\tau)d\tau - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2$$

Let

$$\kappa_r^* \doteq \kappa_r + a\sigma_r^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{jr}(\tau)d\tau$$
$$\kappa_\lambda^* \doteq \kappa_\lambda + a\sigma_\lambda^2\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{j\lambda}(\tau)d\tau$$
$$\kappa_{r\lambda}^* \doteq a\sigma_\lambda^2\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{j\lambda}(\tau)d\tau$$
$$\kappa_{\lambda r}^* \doteq a\sigma_r^2\sum_j\int_0^\infty\left[\Psi_{j\lambda}(\tau)+\Omega_j(\tau)\right]A_{jr}(\tau)d\tau$$

First, I obtain the system of ODEs

$$\begin{aligned}
A'_{gr}(\tau) &= 1 - A_{gr}(\tau)\kappa^*_r - A_{g\lambda}(\tau)\kappa^*_{r\lambda} \\
A'_{cr}(\tau) &= 1 - A_{cr}(\tau)\kappa^*_r - A_{c\lambda}(\tau)\kappa^*_{r\lambda} \\
A'_{g\lambda}(\tau) &= -A_{gr}(\tau)\kappa^*_{\lambda r} - A_{g\lambda}(\tau)\kappa^*_{\lambda} - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 \\
A'_{c\lambda}(\tau) &= 1 - A_{c\lambda}(\tau)\kappa^*_{\lambda} - A_{cr}(\tau)\kappa_{\lambda r} - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2
\end{aligned}$$

which is to be solved with the boundary conditions $A_{gr}(0) = 0$, $A_{cr}(0) = 0$, $A_{g\lambda}(0) = 0$, and $A_{c\lambda}(0) = 0$. Given solutions for $A_{gr}(\tau)$, $A_{cr}(\tau)$, $A_{g\lambda}(\tau)$, and $A_{c\lambda}(\tau)$, I solve for $C'_g(\tau)$ and $C'_c(\tau)$ by matching constant terms. This yields

$$\begin{aligned}
C'_g(\tau) &= \kappa^*_r\bar{r}^* A_{gr}(\tau) + \kappa^*_\lambda\bar{\lambda}^* A_{g\lambda}(\tau) \\
C'_c(\tau) &= \kappa^*_r\bar{r}^* A_{cr}(\tau) + \kappa^*_\lambda\bar{\lambda}^* A_{c\lambda}(\tau)
\end{aligned}$$

where

$$\begin{aligned}
\kappa^*_r\bar{r}^* &\doteq \kappa_r\bar{r} - a\sigma_r^2\sum_j\int_0^\infty \Psi_{j0}(\tau)A_{jr}(\tau)d\tau \\
\kappa^*_\lambda\bar{\lambda}^* &\doteq \kappa_\lambda\bar{\lambda} - a\sigma_\lambda^2\sum_j\int_0^\infty \Psi_{j0}(\tau)A_{j\lambda}(\tau)d\tau
\end{aligned}$$

Integrating with respect to τ completes the proof. ■

C.2.10 Proof of Proposition 9

Proof. Recall that the market price associated to risk factor s is given by

$$\eta_{s,t} = -\sum_j\int_0^\infty\left(\frac{1}{\lambda_t}\left(\Psi_{jr}(\tau)r_t + \Psi_{j\lambda}(\tau)\lambda_t + \Psi_{j0}(\tau)\right) + \Omega_j(\tau)\right)A_{js}(\tau)d\tau$$

Hence, risk prices do not respond to r_t if

$$\sum_j\int_0^\infty \Psi_{jr}(\tau)A_{js}(\tau)d\tau = 0$$

for $s \in \{r, \lambda\}$. This is equivalent to showing that when $\theta_j(\tau) = 0$, $\kappa^*_r = \kappa_r$ and $\kappa^*_{r\lambda} = \kappa_{r\lambda}$. In equilibrium,

$$\begin{aligned}
A'_{gr}(\tau) &= 1 - A_{gr}(\tau)\kappa^*_r - A_{g\lambda}(\tau)\kappa^*_{r\lambda} \\
A'_{cr}(\tau) &= 1 - A_{cr}(\tau)\kappa^*_r - A_{c\lambda}(\tau)\kappa^*_{r\lambda}
\end{aligned}$$

where

$$\kappa_r^* \doteq \kappa_r + a\sigma_r^2 \sum_j \int_0^\infty \Psi_{jr}(\tau) A_{jr}(\tau) d\tau = \kappa_r + I_{rr}$$
$$\kappa_{r\lambda}^* \doteq a\sigma_\lambda^2 \sum_j \int_0^\infty \Psi_{jr}(\tau) A_{j\lambda}(\tau) d\tau = I_{r\lambda}$$

It follows that

$$A'_{gr}(\tau) + A_{gr}(\tau)\kappa_r - 1 = -A_{gr}(\tau) I_{rr} - A_{g\lambda}(\tau) I_{r\lambda}$$
$$A'_{cr}(\tau) + A_{cr}(\tau)\kappa_r - 1 = -A_{cr}(\tau) I_{rr} - A_{c\lambda}(\tau) I_{r\lambda}$$

Using the definition of $\Psi_{gr}(\tau)$ and $\Psi_{cr}(\tau)$ with $\theta_j(\tau) = 0$

$$\begin{aligned}
I_{rr} &= a\sigma_r^2 \int_0^\infty \Psi_{jr}(\tau) A_{jr}(\tau) d\tau \\
&= a\sigma_r^2 \left(\int_0^\infty \left(\frac{A'_{gr} + A_{gr}(\tau)\kappa_r - 1}{a^g \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2\right]} \right) A_{gr}(\tau) d\tau \right. \\
&\qquad \left. + \int_0^\infty \left(\frac{A'_{cr} + A_{cr}(\tau)\kappa_r - 1}{a^c \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2\right]} \right) A_{cr}(\tau) d\tau \right) \\
&= a\sigma_r^2 \left(\int_0^\infty \left(\frac{-A_{gr}(\tau) I_{rr} - A_{g\lambda}(\tau) I_{r\lambda}}{a^g \left[A_{gr}(\tau)^2 \sigma_r^2 + A_{g\lambda}(\tau)^2 \sigma_\lambda^2\right]} \right) A_{gr}(\tau) d\tau \right. \\
&\qquad \left. + \int_0^\infty \left(\frac{-A_{cr}(\tau) I_{rr} - A_{c\lambda}(\tau) I_{r\lambda}}{a^c \left[A_{cr}(\tau)^2 \sigma_r^2 + A_{c\lambda}(\tau)^2 \sigma_\lambda^2\right]} \right) A_{cr}(\tau) d\tau \right) \\
&= -I_{rr} \cdot \left(a\sigma_r^2 \int_0^\infty \frac{A_{gr}(\tau)^2}{\alpha_g(\tau)} + \frac{A_{cr}(\tau)^2}{\alpha_c(\tau)} d\tau \right) - I_{r\lambda} \cdot \left(a\sigma_r^2 \int_0^\infty \frac{A_{gr}(\tau) A_{g\lambda}(\tau)}{\alpha_g(\tau)} + \frac{A_{cr}(\tau) A_{c\lambda}(\tau)}{\alpha_c(\tau)} d\tau \right)
\end{aligned}$$

where

$$\alpha_j(\tau) \doteq a^j \left[A_{jr}(\tau)^2 \sigma_r^2 + A_{j\lambda}(\tau)^2 \sigma_\lambda^2 \right] > 0$$

A similar argument yields

$$I_{r\lambda} = -I_{rr} \cdot \left(a\sigma_\lambda^2 \int_0^\infty \frac{A_{gr}(\tau) A_{g\lambda}(\tau)}{\alpha_g(\tau)} + \frac{A_{cr}(\tau) A_{c\lambda}(\tau)}{\alpha_c(\tau)} d\tau \right) - I_{r\lambda} \cdot \left(a\sigma_\lambda^2 \int_0^\infty \frac{A_{gr}(\tau)^2}{\alpha_g(\tau)} + \frac{A_{cr}(\tau)^2}{\alpha_c(\tau)} d\tau \right)$$

I thus obtain a system of two equations in the two unknowns I_{rr} and $I_{r\lambda}$

$$I_{rr} = -I_{rr} \cdot a\sigma_r^2 \cdot m_r - I_{r\lambda} \cdot a\sigma_r^2 \cdot m_{r\lambda}$$
$$I_{r\lambda} = -I_{rr} \cdot a\sigma_\lambda^2 \cdot m_{r\lambda} - I_{r\lambda} \cdot a\sigma_\lambda^2 \cdot m_\lambda$$

where $m_\lambda > 0$ and $m_r > 0$. The unique solution of the system is $I_{rr} = I_{r\lambda}$, which is the desired result.

Next, I show that when $\delta_j(\tau) = 0$, $\kappa^*_\lambda = \kappa_\lambda$ and $\kappa^*_{\lambda r} = \kappa_{\lambda r}$. In equilibrium, it holds that

$$A'_{g\lambda}(\tau) = -A_{gr}(\tau)\kappa^*_{\lambda r} - A_{g\lambda}(\tau)\kappa^*_\lambda - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2$$
$$A'_{c\lambda}(\tau) = 1 - A_{c\lambda}(\tau)\kappa^*_\lambda - A_{cr}(\tau)\kappa^*_{\lambda r} - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2$$

where

$$\kappa^*_\lambda \doteq \kappa_\lambda + a\sigma_\lambda^2 \sum_j \int_0^\infty \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] A_{j\lambda}(\tau)d\tau = \kappa_\lambda + a\sigma_\lambda^2 \sum_j \int_0^\infty \Psi_{j\lambda}(\tau)A_{j\lambda}(\tau)d\tau$$
$$\kappa^*_{\lambda r} \doteq a\sigma_r^2 \sum_j \int_0^\infty \left[\Psi_{j\lambda}(\tau) + \Omega_j(\tau)\right] A_{jr}(\tau)d\tau = a\sigma_r^2 \sum_j \int_0^\infty \Psi_{j\lambda}(\tau)A_{jr}(\tau)d\tau$$

where the second equality uses that $\Omega_j(\tau) = 0$ when $\delta_j(\tau) = 0$. Using the definition of $\Psi_{j\lambda}(\tau)$ gives

$$\begin{aligned}
I_{\lambda\lambda} &= a\sigma_\lambda^2 \int_0^\infty \left[\frac{A'_{g\lambda}(\tau) + A_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2}{a^g\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]}\right] A_{g\lambda}(\tau)d\tau \\
&\quad + a\sigma_\lambda^2 \int_0^\infty \left[\frac{A'_{c\lambda}(\tau) + A_{c\lambda}(\tau)\kappa_\lambda + \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 + \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2}{a^c\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]}\right] A_{c\lambda}(\tau)d\tau \\
&= a\sigma_\lambda^2 \int_0^\infty \left[\frac{-A_{gr}(\tau)I_{\lambda r} - A_{g\lambda}(\tau)I_{\lambda\lambda}}{a^g\left[A_{gr}(\tau)^2\sigma_r^2 + A_{g\lambda}(\tau)^2\sigma_\lambda^2\right]}\right] A_{g\lambda}(\tau)d\tau \\
&\quad + a\sigma_\lambda^2 \int_0^\infty \left[\frac{-A_{cr}(\tau)I_{\lambda r} - A_{c\lambda}(\tau)I_{\lambda\lambda}}{a^c\left[A_{cr}(\tau)^2\sigma_r^2 + A_{c\lambda}(\tau)^2\sigma_\lambda^2\right]}\right] A_{c\lambda}(\tau)d\tau
\end{aligned}$$

Repeating the same steps for $I_{\lambda r}$ delivers again a system of two equations in the two unknowns $I_{\lambda r}$ and $I_{\lambda\lambda}$. Since the unique solution is $I_{\lambda r} = I_{\lambda\lambda}$, this gives the first result.

It follows that the system of ODEs collapses to

$$\begin{aligned}
A'_{gr}(\tau) &= 1 - A_{gr}(\tau)\kappa_r \\
A'_{cr}(\tau) &= 1 - A_{cr}(\tau)\kappa_r \\
A'_{g\lambda}(\tau) &= -A_{g\lambda}(\tau)\kappa_\lambda - \frac{1}{2}A_{gr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{g\lambda}(\tau)^2\sigma_\lambda^2 \\
A'_{c\lambda}(\tau) &= 1 - A_{c\lambda}(\tau)\kappa_\lambda - \frac{1}{2}A_{cr}(\tau)^2\sigma_r^2 - \frac{1}{2}A_{c\lambda}(\tau)^2\sigma_\lambda^2
\end{aligned}$$

The functions $A'_{gr}(\tau)$ and $A'_{cr}(\tau)$ are identical and equal to

$$A_{cr}(\tau) = A_{cr}(\tau) = \frac{1 - e^{-\kappa_r\tau}}{\kappa_r} > 0$$

since $\kappa_r > 0$, which completes the proof. Note further, for $A'_{g\lambda}(0) = 0$ and $A'_{c\lambda}(0) = 1$. Then

$$\begin{aligned}
A''_{g\lambda}(\tau) &= -A'_{g\lambda}(\tau)\kappa_\lambda - A_{gr}(\tau)A'_{gr}(\tau)\sigma_r^2 - A_{g\lambda}(\tau)A'_{g\lambda}(\tau)\sigma_\lambda^2 \\
A'''_{g\lambda}(\tau) &= -A''_{g\lambda}(\tau)\kappa_\lambda - [A'_{gr}(\tau)^2 + A_{gr}(\tau)A''_{gr}(\tau)]\sigma_r^2 - [A'_{g\lambda}(\tau)^2 + A'_{g\lambda}(\tau)A''_{g\lambda}(\tau)]\sigma_\lambda^2
\end{aligned}$$

so that

$$A''_{g\lambda}(0) = 0$$
$$A'''_{g\lambda}(0) = -\sigma_r^2 < 0$$

■

C.3 Poisson Processes and Idiosyncratic Defaults

Let the increment of the Poisson process N_t be

$$dN_t = \begin{cases} 0 & : \text{wp} \quad 1-\lambda dt \\ 1 & : \text{wp} \quad \lambda dt \\ \geq 2 & : \text{wp} \quad 0 \end{cases}$$

Again, the intuition is that in an interval dt, the probability of two or more jumps goes to zero because $(dt)^k \approx 0$ for $k \geq 2$. Consider a continuum of bonds $i \in [0,1]$. Each of these bonds follows the dynamics

$$\frac{dP_t^i}{P_t^i} = \mu dt + \sigma dW_t + dN_t^i\,(\omega - 1)$$

where ω is the recovery rate. The increment in the point process dN_t^i describes whether bond i defaults or not. I assume that dN_t^i are independent across i, but they have the same intensity λ. The dynamics of the continuum of bonds is

$$\frac{dP_t}{P_t} \doteq \int_0^1 \frac{dP_t^i}{P_t^i} di = \int_0^1 (\mu dt) di + \int_0^1 (\sigma dW_t) di + \int_0^1 dN_t^i(\omega - 1) = \mu dt + \sigma dW_t + (\omega - 1)\int_0^1 dN_t^i$$

The quantity $\int_0^1 dN_t^i$ can be thought of as the cross-sectional average defaults across all bonds. Consider an equally spaced partition of the unit interval $\Pi \doteq \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots \frac{N-1}{N}, 1\right\}$. The norm of the partition is $\sup_n \Pi = \frac{1}{N}$. Hence, $\Pi \to 0$ can be written as $N \to \infty$. As a result

$$\int_0^1 dN_t^i = \lim_{\Pi \to 0} \sum_{k=1}^{N} dN_t^i \cdot (i_{k+1} - i_k) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} dN_t$$

Since $\mathbb{E}[dN_t] = \lambda dt < \infty$ and dN_t^i are i.i.d across bonds, the Law of Large numbers gives

$$\int_0^1 dN_t^i = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} dN_t \overset{\text{LLN}}{=} \mathbb{E}[dN_t] = \lambda dt$$

Therefore

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t + (\omega - 1)\lambda dt$$

and the same argument goes through if λ_t is time-varying but known at t.

C.4 Stochastic Volatility and Time-varying Risk Prices

When shocks to default intensity are heteroscedastic and habitat demand curves have the same form as in Vayanos and Vila (2021), yields are no longer affine in the risk factors s_t. As a result, the exponentially-affine conjecture breaks, hinting at a pricing function with a different functional form. To show why, I consider a simplified version o in which arbitrageurs only invest in the short term rate and in corporate bonds. I omit the security index j and I abstract for demand shocks. The decision problem of the arbitrageurs and the specification of habitat demand is the same as in Section 2. However, I assume that default intensity has square root dynamics of the form

$$d\lambda_t = \kappa_\lambda(\overline{\lambda} - \lambda_t) + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}$$

Square-root dynamics ensure that $\lambda_t > 0$ and introduce heteroscedasticity. I conjecture that

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)]}$$

Following the same steps implies that the arbitrageurs' first-order condition is

$$\mu_t^{(\tau)} - r_t = \lambda_t - A_r(\tau)\sigma_r \cdot \eta_{r,t} - A_\lambda(\tau)\sigma_\lambda \cdot \eta_{\lambda,t}$$

where the market price of default intensity risk is

$$\pi_{\lambda,t} \doteq -\sigma_\lambda \lambda_t \left(\int_0^\infty x_t^{(\tau)} A_\lambda(\tau) d\tau \right)$$

Market clearing requires $x_t^{(\tau)} + z_t^{(\tau)}$, so that

$$x_t^{(\tau)} = \theta(\tau) - \alpha(\tau) \left[A_r(\tau) r_t + A_\lambda(\tau)\lambda_t + C(\tau) \right]$$

After substituting the market clearing condition into the market prices of risk, I obtain

$$\eta_{\lambda,t} = -\sigma_\lambda \lambda_t \left(\int_0^\infty \{ \theta(\tau) - \alpha(\tau) \left[A_r(\tau) r_t + A_\lambda(\tau)\lambda_t + C(\tau) \right] \} A_\lambda(\tau) d\tau \right) \tag{2}$$

Stochastic volatility introduces a second source of variation in risk premia. As a result, the right-hand side of the arbitrageurs' first-order condition includes a product of two affine functions, whereas the Itô term on the left-hand side remains linear in the state variables. Matching coefficients on λ_t^2 implies

$$0 = -A_\lambda(\tau) a \sigma_\lambda^2 \int_0^\infty \alpha(\tau) A_r(\tau) A_\lambda(\tau) d\tau$$

This only holds provided that $A_\lambda(\tau) = 0$ for all τ, which leads to contradiction.

D Additional Results

D.1 Bond Return Predictability

(a) Treasury excess returns and EBP.

(b) Treasury excess returns, EBP and controls

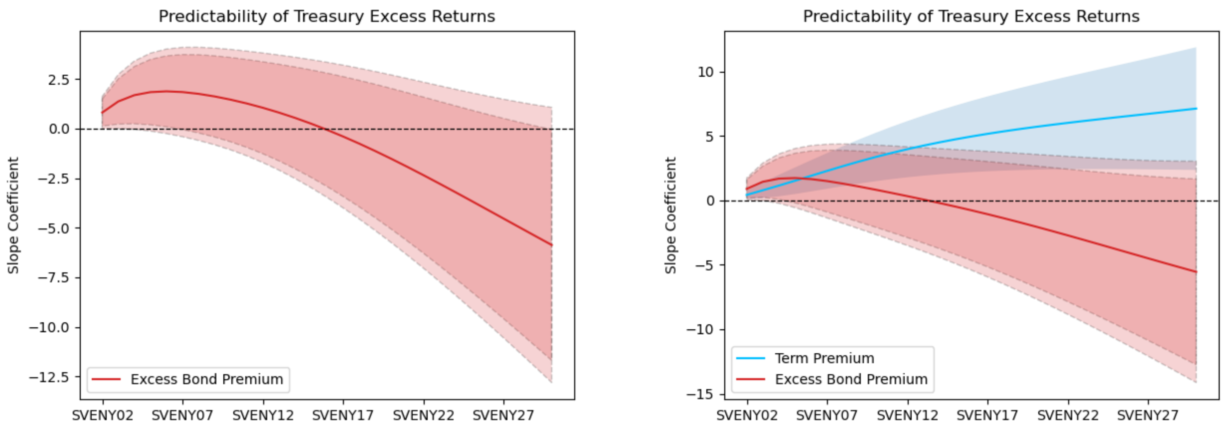

Figure 14: Parameter estimates of regression (27) and associated confidence intervals. The left panel presents regressions of bond excess returns on the excess bond premium (EBP) (Gilchrist & Zakrajšek, 2012). The right panel presents regressions of bond excess returns on the excess bond premium controlling for the Treasury term premium (Adrian et al., 2013) and the VIX. Shaded areas represent 90% and 95% confidence intervals constructed using Hodrick (1992) standard errors. The monthly sample is January 1990 to April 2024.