ONLINE APPENDIX

Constrained Efficiency in a Human Capital Model

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Abstract

This document contains supplemental materials for "Constrained Efficiency in a Human Capital Model ". We discuss the implementation of the constrained efficient allocation, formally analyze the effects of consumption poor household's relative income sources on pecuniary externalities, provide a further sensitivity analysis, and discuss the effects of the time investment model and the effects of different market structure. We also provide the proof of propositions in the main text and formally derive the planner's first-order conditions in an infinite horizon model.

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A Implementation of the constrained efficient allocation

In this section, we discuss how we can implement the constrained efficient allocation. The constrained efficient allocation can be attained if the government directly modifies each consumer's savings and human capital investments. We can also implement the constrained efficient allocation as a competitive equilibrium using a tax(subsidy)-transfer(lump-sum tax) system that is history dependent and induces no reallocation of income across individuals or realization of idiosyncratic shocks.

In the two-period model with additive idiosyncratic shock, history-dependent taxes and transfers are represented by the initial heterogeneity-dependent taxes and transfers. The tax(subsidy)-transfer(lump-sum tax) system is then completely characterized by the linear proportional tax rates of capital income and labor income $(\tau_k(k,h), \tau_l(k,h))$ and transfers $(T(k,h,e))$, which are functions of initial wealth and human capital (k,h) and idiosyncratic shock realization e.

Proposition 5. *There exists a triple of history-dependent capital income tax rates, labor income tax rates, and income transfers $(\tau_k(k,h), \tau_l(k,h), T(k,h,e))$ that implements the constrained planner's allocation.*

Proof By setting capital income tax rates and labor income tax rates in the following way, the Euler equation of the competitive equilibrium and the constrained planner's problem become equivalent:

$$\tau_k(k,h) = -\frac{\Delta_k}{(F_k - \delta)E_s[u'(c_1)|k,h]} \tag{1}$$

$$\tau_l(k,h) = -\frac{\Delta_h}{f_H E[u'(c_1)|k,h]} \tag{2}$$

Then, the following transfer function, which imposes no income transfer across agents, implements the constrained efficient allocation.

$$T(k,h,e) = \tau_k r k'(k,h) + \tau_l w[h'(k,h) + e]$$

■

Or alternatively, we can implement the same allocation using linear tax rates[1] of capital income and linear subsidy rates of human capital $(\tau_k(k,h), s(k,h))$, lumpsum taxes in period 0 $T_0(k,h)$, and lumpsum transfers in period 1 $T_1(k,h)$, such that $\tau_k(k,h)$ is set as in (1) and $s(k,h)$ is set to

$$s(k,h) = \frac{\beta g_{x_h}\Delta_h}{u'(c_0(c_0(k,h)))}.$$

[1] That is, with subsidy, the cost of investing x_h in human capital is $(1 - s(k,h))x_h(k,h)$

B Formal Analysis of Section 3.1 and Supplement of Section 3.2

B.1 Effects of consumption poor households' relative income sources

In section 3.1 of the main text, we discussed the importance of relative income of the consumption poor. We formally show its importance in this section. First, we investigate the importance of consumption poor households' relative sources of income — human wealth vs. nonhuman wealth. From the baseline distribution, we now make two different ϵ-perturbations: 1. ϵ-perturbation around h_1, and 2. ϵ-perturbation around h_2. The perturbations have the following forms:

$[Case1]$

$$\gamma_1(k',h') = \begin{cases} \frac{1}{4}\rho & \text{if } (k',h') = (k_1, h_1^{-\epsilon}) \\ \frac{1}{4}\rho & \text{if } (k',h') = (k_1, h_1^{+\epsilon}) \\ \frac{1}{2}(1-\rho) & \text{if } (k',h') = (k_1, h_2) \\ \frac{1}{4}(1-\rho) & \text{if } (k',h') = (k_2, h_1^{-\epsilon}) \\ \frac{1}{4}(1-\rho) & \text{if } (k',h') = (k_2, h_1^{+\epsilon}) \\ \frac{1}{2}\rho & \text{if } (k',h') = (k_2, h_2), \end{cases}$$

where

$h_1^{-\epsilon} = (1-\theta_h(1+\epsilon)), \quad h_1^{+\epsilon} = (1-\theta_h(1-\epsilon))$

$[Case2]$

$$\gamma_1(k',h') = \begin{cases} \frac{1}{2}\rho & \text{if } (k',h') = (k_1, h_1) \\ \frac{1}{4}(1-\rho) & \text{if } (k',h') = (k_1, h_2^{-\epsilon}) \\ \frac{1}{4}(1-\rho) & \text{if } (k',h') = (k_1, h_2^{+\epsilon}) \\ \frac{1}{2}(1-\rho) & \text{if } (k',h') = (k_2, h_1) \\ \frac{1}{4}\rho & \text{if } (k',h') = (k_2, h_2^{-\epsilon}) \\ \frac{1}{4}\rho & \text{if } (k',h') = (k_2, h_2^{+\epsilon}), \end{cases}$$

where

$h_2^{-\epsilon} = (1+\theta_h(1-\epsilon)), \quad h_2^{+\epsilon} = (1+\theta_h(1+\epsilon))$

Both case 1 and case 2 have the same mean and variance of k' and h', but the perturbation in case 1 makes the consumption poor household have less human capital compared to the case 2 perturbation. The next proposition shows that in an economy where the consumption poor have relatively less human capital, $\Delta_k|_{CE}$ is relatively lower, even though the measure of inequality of wealth and human capital does not change.

Proposition 6. *Suppose that* $\epsilon > \theta_k - \theta_h$. *If* $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$ *or* $\rho \geq \frac{1}{2}$, $\Delta_k|_{CE}^{case1} < \Delta_k|_{CE}^{case2}$.

Proof We need to show that $\Delta_k|_{CE}^{case1} - \Delta_k|_{CE}^{case1} < 0$. Using a CARA-normal specification,

$$\begin{aligned} \Delta_k|_{CE}^{case1} - \Delta_k|_{CE}^{case1} &= -\psi F_H F_{HK} \sigma_e^2 \beta \exp(\frac{\psi^2}{2} w^2 \sigma_e^2)[\tilde{\Delta}_{k,1}^{case1} - \tilde{\Delta}_{k,1}^{case2}] \\ &\quad + \exp(\frac{\psi^2}{2} w^2 \sigma_e^2) F_{KK} K \beta [\tilde{\Delta}_{k,2}^{case1} - \tilde{\Delta}_{k,2}^{case2}] \end{aligned}$$

where

$$\begin{aligned}
\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} \quad = \quad & \frac{\rho}{2}\left[\frac{1}{2}u'(wh_1^{-\epsilon}+rk_1)+\frac{1}{2}u'(wh_1^{+\epsilon}+rk_1)\right]+\frac{1-\rho}{2}u'(wh_2+rk_1) \\
& +\frac{1-\rho}{2}\left[\frac{1}{2}u'(wh_1^{-\epsilon}+rk_2)+\frac{1}{2}u'(wh_1^{+\epsilon}+rk_2)\right]+\frac{\rho}{2}u'(wh_2+rk_2) \\
& -\frac{\rho}{2}u'(wh_1+rk_1)-\frac{1-\rho}{2}\left[\frac{1}{2}u'(wh_2^{-\epsilon}+rk_1)+\frac{1}{2}u'(wh_2^{+\epsilon}+rk_1)\right] \\
& \frac{1-\rho}{2}u'(wh_1^{-\epsilon}+rk_2)-\frac{1-\rho}{2}\left[\frac{1}{2}u'(wh_2^{-\epsilon}+rk_2)+\frac{1}{2}u'(wh_2^{+\epsilon}+rk_2)\right] \\
\tilde{\Delta}^{case1}_{k,2} - \tilde{\Delta}^{case2}_{k,2} \quad = \quad & \frac{\rho}{2}\left[\frac{1}{2}u'(wh_1^{-\epsilon}+rk_1)(\frac{k_1}{K}-\frac{h_1^{-\epsilon}}{H})+\frac{1}{2}u'(wh_1^{+\epsilon}+rk_1)(\frac{k_1}{K}-\frac{h_1^{+\epsilon}}{H})\right]+\frac{1-\rho}{2}u'(wh_2+rk_1)(\frac{k_1}{K}-\frac{h_2}{H}) \\
& +\frac{1-\rho}{2}\left[\frac{1}{2}u'(wh_1^{-\epsilon}+rk_2)(\frac{k_2}{K}-\frac{h_1^{-\epsilon}}{H})+\frac{1}{2}u'(wh_1^{+\epsilon}+rk_2)(\frac{k_2}{K}-\frac{h_1^{+\epsilon}}{H})\right]+\frac{\rho}{2}u'(wh_2+rk_2)(\frac{k_2}{K}-\frac{h_1}{H}) \\
& -\frac{\rho}{2}u'(wh_1+rk_1)(\frac{k_1}{K}-\frac{h_1}{H})-\frac{1-\rho}{2}\left[\frac{1}{2}u'(wh_2^{-\epsilon}+rk_1)(\frac{k_1}{K}-\frac{h_2^{-\epsilon}}{H})+\frac{1}{2}u'(wh_2^{+\epsilon}+rk_1)(\frac{k_1}{K}-\frac{h_2^{+\epsilon}}{H})\right] \\
& -\frac{1-\rho}{2}u'(wh_1+rk_2)(\frac{k_2}{K}-\frac{h_1}{H})-\frac{\rho}{2}\left[\frac{1}{2}u'(wh_2^{-\epsilon}+rk_2)(\frac{k_2}{K}-\frac{h_2^{-\epsilon}}{H})+\frac{1}{2}u'(wh_2^{+\epsilon}+rk_2)(\frac{k_2}{K}-\frac{h_2^{+\epsilon}}{H})\right]
\end{aligned}$$

We want to show $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} > 0$ and $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} > 0$. First, $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1}$ can be rewritten as follows.

$$\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} \quad = \quad \frac{\rho}{2}A+\frac{1-\rho}{2}B,$$

where

$$A=\left\{\frac{1}{2}\left[u'(wh_1^{-\epsilon}+rk_1)+u'(wh_1^{+\epsilon}+rk_1)\right]-u'(wh_1+rk_1)\right\}-\left\{\frac{1}{2}\left[u'(wh_2^{-\epsilon}+rk_2)+u'(wh_2^{+\epsilon}+rk_2)\right]-u'(wh_2+rk_2)\right\}$$
$$B=\left\{\frac{1}{2}\left[u'(wh_1^{-\epsilon}+rk_2)+u'(wh_1^{+\epsilon}+rk_2)\right]-u'(wh_1+rk_2)\right\}-\left\{\frac{1}{2}\left[u'(wh_2^{-\epsilon}+rk_1)+u'(wh_2^{+\epsilon}+rk_1)\right]-u'(wh_2+rk_1)\right\}.$$

Since u' is decreasing and convex and $wh_1+rk_1 < wh_2+rk_2$, $A>0$. Similarly, if $wh_1+rk_2 \leq wh_2+rk_1$ then $B \geq 0$. If $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, then $wh_1'+rk_2' \leq wh_2'+rk_1'$, by the following.

$$\begin{aligned}
& (wh_1'+rk_2')-(wh_2'+rk_1') \\
= \quad & \left((1-\alpha)K^\alpha H^{1-\alpha}\frac{h_1'}{H}+\alpha K^\alpha H^{1-\alpha}\frac{k_2'}{K}-\delta_k k_2'\right)-\left((1-\alpha)K^\alpha H^{1-\alpha}\frac{h_2'}{H}+\alpha K^\alpha H^{1-\alpha}\frac{k_1'}{K}-\delta_k k_1'\right) \\
= \quad & K^\alpha H^{1-\alpha}\left[((1-\alpha)(1-\theta_h)+\alpha(1-\theta_k))-((1-\alpha)(1+\theta_h)+\alpha(1-\theta_k))\right]-\delta_k(k_2'-k_1') \\
< \quad & 2K^\alpha H^{1-\alpha}\left[\alpha\theta_k-(1-\alpha)\theta_h\right] \leq 0
\end{aligned}$$

Thus if $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} > 0$.

In the other case, despite negative B, $A+B>0$ since u' is decreasing and convex ($wh_1+rk_1 <$

$wh_2 + rk_1$ and $wh_1 + rk_2 < wh_2 + rk_2$). Thus if $\rho \geq \frac{1}{2}$ then $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} > 0$ because

$$\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} = \frac{1-\rho}{2}(A+B) + \frac{2\rho - 1}{2}A > 0.$$

Next, $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1}$ can be written as follows after some algebra.

$$\begin{aligned} \tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} &= \frac{\rho}{2}\left[\{(\delta_{11} - \delta_{22}) - (X_{11} + X_{22})\}(\theta_k - \theta_h) + (\delta_{11} - \delta_{22})(\epsilon - (\theta_k - \theta_h))\right] \\ &\quad + \frac{1-\rho}{2}[(\delta_{12} - \delta_{21})\epsilon + (X_{12} + X_{21})(\theta_k + \theta_h)] \end{aligned} \tag{3}$$

where

$$\begin{aligned} X_{ij} &= \frac{1}{2}[u'(wh_i^{-\epsilon} + rk_j) + u'(wh_i^{+\epsilon} + rk_j)] - u'(wh_i + rk_j) \\ \delta_{ij} &= \frac{1}{2}[u'(wh_i^{-\epsilon} + rk_j) - u'(wh_i^{+\epsilon} + rk_j)] \end{aligned}$$

The first term in the first bracket is positive because

$$\begin{aligned} (\delta_{11} - \delta_{22}) - (X_{11} + X_{22}) &= (\delta_{11} - X_{11}) - (\delta_{22} + X_{22}) \\ &= [u'(wh_1 + rk_1) - u'(wh_1^{+\epsilon} + rk_1)] - [u'(wh_2^{-\epsilon} + rk_2) - u'(wh_2 + rk_2)] > 0. \end{aligned}$$

From (3), the second term in the first bracket is also positive because $\delta_{11} - \delta_{22} > 0$ and by assumption $\epsilon > \theta_k - \theta_h$. The second term in the second bracket is positive because $X_{ij} > 0$ for all i, j and thus we only need to show that the first term in the second bracket is nonnegative. If $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, then $wh_1' + rk_2' \leq wh_2' + rk_1'$ and thus $\delta_{12} - \delta_{21} \geq 0$.

On the other case ($\frac{\theta_k}{\theta_h} > \frac{1-\alpha}{\alpha}$), $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1}$ is still positive if $\rho \geq \frac{1}{2}$. This is because $\tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1}$ can be rewritten as

$$\begin{aligned} \tilde{\Delta}^{case1}_{k,1} - \tilde{\Delta}^{case2}_{k,1} &= \frac{1-\rho}{2}\begin{bmatrix} \{(\delta_{11} - X_{11}) - (\delta_{21} - X_{21}) + (\delta_{12} + X_{12}) - (\delta_{22} + X_{22})\}(\theta_k - \theta_h) \\ +\{(\delta_{11} - \delta_{22}) + (\delta_{12} - \delta_{21})\}(\epsilon - (\theta_k - \theta_h)) \end{bmatrix} \\ &\quad + \frac{2\rho - 1}{2}[(\delta_{11} - \delta_{22})\epsilon - (X_{11} + X_{22})(\theta_k - \theta_h)] \end{aligned}$$

and we can easily show that every bracket is positive.

In sum, $\Delta_k|^{case1}_{CE} - \Delta_k|^{case1}_{CE} < 0$ if either $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$ or $\rho \geq \frac{1}{2}$. ■

Figure 1: Effects of correlation on $\Delta_{k1}|_{CE}$

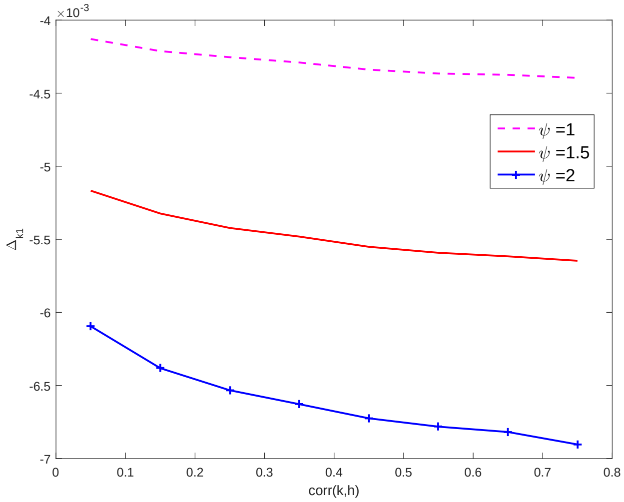
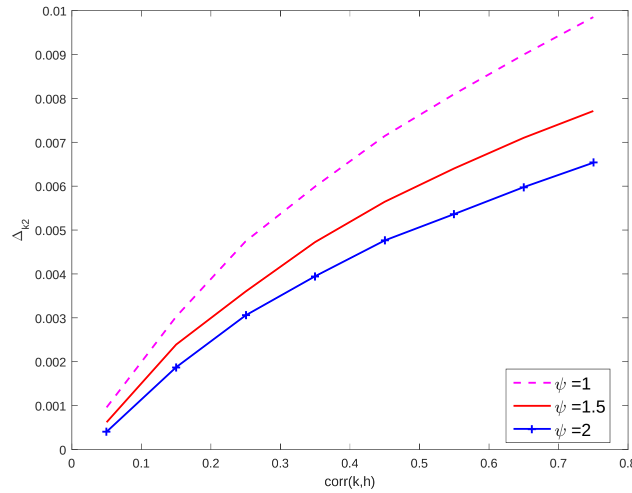


Figure 2: Effects of correlation on $\Delta_{k2}|_{CE}$



B.2 Supplementary Graphs for Section 3.2

Figure 1 and Figure 2 show the effects of correlation on $\Delta_{k1}|_{CE}$ and $\Delta_{k2}|_{CE}$ with three different coefficient of risk aversion ψ, using the numerical example in section 3.2.

C Infinite Horizon Model: Competitive Equilibrium and Constrained Efficient Allocation

In the main text, to save space, we did not define a competitive equilibrium and derive the planner's first order conditions. In this section, we provide the formal definition and proposition for the first order conditions.

Definition 7. *A recursive competitive equilibrium is a value function, $v(\phi, k, h; A)$, policy functions $c(\Phi, k, h; A)$, $k'(\Phi, k, h; A)$, $x_h(\Phi, k, h; A)$, prices $r(\Phi)$, $w(\Phi)$, and an aggregate law of motion, F s.t.*

1. *Given w, r, and F, policy functions c, k', x_h solve the household's problem and the associated value function is v.*

2. *The price functions satisfy $r(\Phi) = f_K(K, L)$, and $w(\Phi) = f_L(K, L)$, where $K = \int k d\Phi$ and $L = \int h d\Phi$.*

3. *$F(\Phi) = T(\Phi, Q)$*

A steady state for this economy is an invariant distribution $\tilde{\Phi}$, which is the fixed point of the updating operator T.[2]

The following proposition provides the first-order conditions of the constrained planner's problem, which provides the characterization of the constrained-efficient allocation.

Proposition 8. *If the distribution Φ admits a density, the first-order necessary conditions of problem (13) are the following functional equation in the decision rule k'^*, ${x_h}^*$: For all $\{k,h,A\} \in S$,*

1. *First-order condition with respect to k'*

$$
\begin{aligned}
&u'\Big(k(1+r(\Phi)) + hw(\Phi) - k'^*(\Phi,k,h;A) - {x_h}^*(\Phi,k,h;A)\Big) \\
&\quad \geq \beta(1+r(\Phi'))E_{\eta'}\left[u'\left(c^*(\Phi,\Phi',k,h,\eta';A)\right)\right] + \Delta_k, \quad \textit{where} \\
&c^*(\Phi,\Phi',k,h,\eta';A) = k'^*(\Phi,k,h;A)(1+r(\Phi')) + \eta' g(h, {x_h}^*(\Phi,k,h;A),A)w(\Phi') \\
&\qquad\qquad -k'^*(\Phi', k'^*(\Phi,k,h;A), \eta' g(h,{x_h}^*(\Phi,k,h;A),A);A) \\
&\qquad\qquad -x_h^*(\Phi', k'^*(\Phi,k,h;A), \eta' g(h,{x_h}^*(\Phi,k,h;A),A);A), \\
&\Delta_k = \beta\int_S u'\left(k'(1+r(\Phi')) + h'w(\Phi) - k' - x_h'\right)\left[k' f_{KK}(K',L') - h' f_{LK}(K',L')\right] d\Phi',
\end{aligned}
$$

where the inequality becomes an equality if $k'^*(\Phi,k,h;A) > 0$

2. *First-order condition with respect to x_h*

$$
\begin{aligned}
&u'\Big(k(1+r(\Phi))) + hw(\Phi) - k'^*(\Phi,k,h;A) - x_h^*(\Phi,k,h;A)\Big) \\
&= g_{x_h}(h,{x_h}^*(\Phi,k,h;A),A)\beta E_{\eta'}\eta' u'(c^*(\Phi,\Phi',k,h,\eta';A)) \\
&\times\left\{w(\Phi') + \frac{g_h(\eta' g(h,x_h^*(\Phi,k,h;A),A), x_h^*(\Phi', k'^*(\Phi,k,h;A), \eta' g(h,x_h^*(\Phi,k,h;A),A);A),A)}{g_{x_h}(\eta' g(h,x_h^*(\Phi,k,h;A),A), x_h^*(\Phi', k'^*(\Phi,k,h;A), \eta' g(h,x_h^*(\Phi,k,h;A),A);A),A)}\right\} \\
&+g_{x_h}(h,{x_h}^*(\Phi,k,h;A),A)\Delta_h,
\end{aligned}
$$

3. $\Delta_h = -\frac{K}{H}\Delta_k$

Proof See the Appendix G.2 ■

D Robustness with respect to (σ_η, σ_A)

In the main text, we did some sensitivity analysis to see the robustness of our quantitative result. In this section, we support the robustness by performing another sensitivity analysis with respect to

[2] Even though the production technology is constant returns to scale in human capital that is accumulated over time, there is a steady state because, at the individual level, the household faces a diminishing marginal product of human capital ($\phi < 1$).

Table 1: Sensitivity analysis

		σ_η		
		0.111	0.12	0.13
σ_A (recalibrated)		0.148	0.134	0.117
Optimal	Capital-labor ratio	15.04	15.03	15.08
	Interest rate	-1.65%	-1.65%	-1.66%
Optimal	bottom 20% $\frac{k}{K}/\frac{h}{H}$	0.012	0.016	0.017
	$corr(k,h)$	0.61	0.58	0.56
CE	bottom 20% $\frac{k}{K}/\frac{h}{H}$	0.014	0.012	0.011
	$corr(k,h)$	0.75	0.75	0.76

the parameters that impact human capital inequality — the standard deviation of the idiosyncratic shock σ_η and the standard deviation of the learning ability σ_A.

Table 1 shows the results for various combinations of (σ_η, σ_A). For each σ_η, we recalibrated σ_A to generate earnings Gini 0.60. Since we also recalibrated β to generate the interest rate 4%, the capital-labor ratio in a competitive equilibrium is identical to that in the baseline model.

As we can see from the Table 1, varying the variance of shock (σ_η) is not crucial for the quantitative result as long as the model is calibrated to match the earnings Gini. This is because of the following reasons. Since the human capital inequality is generated by the heterogeneity in the learning ability and the shock to human capital, when we increase the variance of the shock, the learning ability inequality should be reduced to match the same earnings Gini. Lower A-inequality makes the redistribution channel through human capital weaker (higher $\Delta_{k2,H}$), while the insurance channel becomes stronger (lower Δ_{k1}). As a result, the two effects offset each other, and the optimal capital-labor ratio is not very sensitive to varying σ_η.

E Time Investment Vs. Money Investment in an Infinite Horizon Model

In section 2.2 of the main text, we already discussed the different implications of the time investment and money investment models using the two-period model. In this section we briefly analyze the time investment model in an infinite horizon model.

Consider the following version of a time investment model as in Huggett et al. (2011). Households have 1 unit of time every period and allocate time between human capital investment (s) and labor supply $(1-s)$. A household with learning ability A that has human capital h_t in period t and invests time s_t will have human capital in period $t+1$, $h_{t+1} = \eta_{t+1} g(h_t, s_t, A) = \eta_{t+1}\{(1-\delta_h)h_t + A(s_t h_t)^\phi\}$, where η_{t+1} is an idiosyncratic shock to human capital. The period budget constraint of the household is $c_t + k_{t+1} = k_t(1+r) + w(1-s_t)h_t$, where $(1-s_t)h_t$ is an effective unit of labor supply.

First, we analyze the impact of the time investment model on Δ_k. The Euler equation of the constrained planner takes exactly the same form as in the money investment model with a slightly different Δ_k, which can be decomposed into:

$$\begin{aligned}\Delta_k &= -F_{KK}K \int\int u'(c')\left[\eta' - 1\right] f(\eta')d\eta' d\Phi \qquad (4)\\ &+F_{KK}K\int\int u'(c')\left[\frac{k'}{K}-1\right]f(\eta')d\eta' d\Phi + F_{KK}K\left\{\begin{array}{l}\int\int u'(c')\left[1-\frac{(1-s')}{L/H}\right]\frac{\eta' g(h,s,A)}{H}f(\eta')d\eta' d\Phi\\ +\int\int u'(c')\left[1-\frac{g(A,h,s)}{H}\right]\eta' f(\eta')d\eta' d\Phi\end{array}\right\}.\end{aligned}$$

With a general utility function, we cannot do an exact additive decomposition, but a rough decomposition in (4) shows that there is an additional channel through the labor supply dispersion in the time investment model — the first integral in the third term of (4). This is because there is an endogenous labor supply dispersion in the time investment model, but this redistribution channel through the labor supply dispersion did not show up in the two-period model, because period 1 is the last period with full labor supply ($s = 1$). With the new labor supply channel, if the consumption poor household works more than the consumption rich household due to the income effect, then this labor dispersion channel is likely to increase Δ_k compared to the money investment model. Thus, time investment might weaken the endogenous human capital's implication for the overaccumulation of capital.

Note that the relationship between Δ_h and Δ_k remains in the time investment model: $\Delta_h = -\frac{K}{H}\Delta_k$. Thus, most of the analytical results in the money investment model still apply.

Next, we discuss that the signs of Δ_k and Δ_h are not sufficient to evaluate overinvestment in human capital. We already discussed this in section 2.2 of the main text, however, in an infinite-horizon model, the implication of the sings for under-/over-investment in human capital is even more involved. We can see this from the planner's first-order condition with respect to s :

$$whu'(c) = g_s w\beta E\left[\eta'\left\{(1-s') + \frac{g'_h h'}{g'_s}\right\}u'(c')\right] + \left\{-h + \beta g_s E\eta'\left[(1-s') + \frac{g'_h h'}{g'_s}\right]\right\}\Delta_h.$$

The second term on the right-hand side is an additional term arising from the pecuniary externalities. Suppose that Δ_h is positive. Even if Δ_h is positive, increasing s has costs and benefits. First, increasing s decreases the labor supply in the current period by h, which creates cost $-h\Delta_h$. Second, increasing s increases human capital and thus the labor supply in the next period by $g_s E[\eta'(1-s')]$ and thus generates benefit $\beta g_s E[\eta'(1-s')]\Delta_h$. Lastly, increasing s decreases s' and thus increases the labor supply in the next period $(1-s')$ by $g_s E[\eta' \frac{g_{h'}h'}{g_{s'}}]$ and thus generates additional benefits $\beta g_s E[\eta' \frac{g_{h'}h'}{g_{s'}}]\Delta_h$. Thus, we need to compare the cost of decreasing working time $(1-s)$ and the benefit of increasing the effective labor supply $(h'(1-s'))$, to determine whether there is over-/under-investment in human capital.

In sum, in the time investment model, the implication of the pecuniary externalities requires more investigation of the effect of the labor supply dispersion and the relationship between s and aggregate L (in the steady state), which is left for future study. However, we would like to emphasize that the redistribution channel through human capital inequality still exists and its implication for the effective labor supply is exactly the same as that in the money investment model.

F Extension: Different Market Structure

In this paper, we analyzed the pecuniary externalities in a standard incomplete market model where the market is incomplete because there is no state-contingent asset. With this market structure, we could show that introducing human capital dispersion can reduce the marginal benefit of saving through pecuniary externalities and thus lower the optimal capital-labor ratio. The pecuniary externalities can exist in an economy where the market is incomplete for endogenous reasons. However, the sources of pecuniary externalities and their implication depend on market structure. Although analyzing pecuniary externalities in different market structures is beyond the scope of this paper, we briefly discuss the effects of introducing human capital in other market structures. As an example, we consider an endogenous incomplete market due to limited commitment and show that introducing human capital can affect the pecuniary externalities in the same directions but for very different reasons.

Consider a limited commitment economy where the amount of asset trading is endogenously restricted because agents can default on the debt contract, as in Kehoe and Levine (1993). The asset market is complete with a fully state-contingent asset, but the following enforcement constraint should be satisfied :

$$\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\pi(s^\tau|s^t)u(c_\tau(s^\tau)) \geq V^{Aut}(h_t, s_t; w_t),$$

where $V^{Aut}(h_t, s_t; w_t)$ is the value of agents who default in period t with human capital stock h_t after realization of a shock s_t. Once an agent defaults, then she can never access the financial market again and thus should live in financial autarky. However, human capital investment cannot be restricted even after deviation and thus the value of deviation is the following :

$$\begin{aligned} V^{Aut}(h_t, s_t; w_t) \quad &= \quad \max_{c(s^t), x(s^t)} \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\pi(s^\tau|s^t)u(c(s^t)) \\ s.t. \qquad & c(s^t) + x(s^t) = w_t f(h_t, s_t), \\ & h_{t+1}(s^t) = g(h_t(s^{t-1}), x_{h,t}(s^t)). \end{aligned}$$

In a limited commitment literature, it is well known that a competitive equilibrium is not constrained efficient because of pecuniary externalities even though both households and the planner are facing the same participation constraints (Kehoe and Perri (2004), Abraham and Carceles-Poveda (2006), and Park (2014)). Compared to the household's Euler equation, the planner's Euler equation has an additional term, exactly because of the pecuniary externalities through autarky prices. Increasing aggregate capital increases the wage rate in autarky w_t, which tightens the enforcement constraint by increasing the value of deviation. The planner wants to tax capital income to internalize these pecuniary externalities, and the externalities become stronger when we introduce human capital. Human capital cannot be taken away when people are deviating and thus the value of deviation $V^{Aut}(h_t, s_t; w_t)$ is more sensitive to the wage rate when there is human capital.

Thus, we can see that even in an endogenously incomplete market due to limited commitment, adding human capital tends to decrease the optimal capital-labor ratio from the perspective of the planner who wants to improve the pecuniary externalities. However, the sources of the pecuniary externalities are very different across models, and thus analyzing the effects of introducing human capital with different market structures will be an interesting question for future research.

G Proof of the Main Text

G.1 Proof of Proposition 4

We first rewrite the $\Delta_{k1}|_{CE}$ and $\Delta_{k2}|_{CE}$, using some simple algebra.

$$\begin{aligned}
\Delta_{k1}|_{CE} &= -\psi F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \int_{k',h'} u'(wh' + rk') \Gamma_1(dk', dh') \\
&= -\psi F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \left[\begin{array}{c} \frac{\rho}{2} u'(wh_1 + rk_1) + \frac{1-\rho}{2} u'(wh_2 + rk_1) \\ + \frac{1-\rho}{2} u'(wh_1 + rk_2) + \frac{\rho}{2} u'(wh_2 + rk_2) \end{array} \right], \\
\Delta_{k2}|_{CE} &= F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \int_{k',h'} u'(wh' + rk') \left[\frac{k'}{K} - \frac{h'}{H} \right] \Gamma_1(dk', dh') \\
&= F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \left[\begin{array}{c} \frac{\rho}{2} u'(wh_1 + rk_1) \left[\frac{k_1}{K} - \frac{h_1}{H} \right] + \frac{1-\rho}{2} u'(wh_2 + rk_1) \left[\frac{k_1}{K} - \frac{h_2}{H} \right] \\ + \frac{1-\rho}{2} u'(wh_1 + rk_2) \left[\frac{k_2}{K} - \frac{h_1}{H} \right] + \frac{\rho}{2} u'(wh_2 + rk_2) \left[\frac{k_2}{K} - \frac{h_2}{H} \right] \end{array} \right].
\end{aligned}$$

(i) By differentiating $\Delta_{k1}|_{CE}$ with respect to ρ, we get
$$\frac{\partial \Delta_{k1}|_{CE}}{\partial \rho} = -\frac{\psi}{2} F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \left[\begin{array}{c} (u'(wh_1 + rk_1) - u'(wh_2 + rk_1)) \\ - (u'(wh_1 + rk_2) - u'(wh_2 + rk_2)) \end{array} \right] < 0,$$
where the last inequality holds because u' is decreasing ($u'' < 0$) and convex ($u''' > 0$).

(ii) By differentiating $\Delta_{k2}|_{CE}$ with respect to ρ, we get
$$\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} = \frac{1}{2} F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \left[\begin{array}{c} (\theta_k - \theta_h)(u'(wh_2 + rk_2) - u'(wh_1 + rk_1)) \\ -(\theta_k + \theta_h)(u'(wh_1 + rk_2) - u'(wh_2 + rk_1)) \end{array} \right].$$

Since $\theta_k > \theta_h$, $wh_2 + rk_2 > wh_1 + rk_1$, the first term in the bracket is negative. If $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, then $wh_1 + rk_2 < wh_2 + rk_1$, as shown in the proof of Proposition 6. Thus, $u'(wh_1 + rk_2) - u'(wh_2 + rk_1) > 0$, and the second term in the bracket is also negative. Since $F_{KK} < 0$ and the whole bracket is negative, $\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} > 0$

G.2 Proof of Proposition 8

Proof We prove the proposition based on a sequential formulation of the constrained planner's problem. We denote a density of the distribution of capital, human capital and learning ability at period t by ψ_t. Then, given density ψ_t, a sequence of the policy rule $\{k'_t(\psi_t, k, h; A), x_{ht}(\psi_t, k, h; A)\}$ of the constrained planner's problem must solve

$$
\begin{aligned}
\max_{c_t, k'_t, x_{ht}} \quad & \sum_t \beta^t \int_S u(c_t)\psi_t(k, h, A)dS \\
s.t. \quad & c_t + k'_t(\psi_t, k, h; A) + x_{ht}(\psi_t, k, h; A) = k[1 + f_K(K(\psi_t), L(\psi_t)) - \delta_k] + h f_L(K(\psi_t), L(\psi_t)) \\
& h_{t+1} = \eta' g(h, x_{ht}(\psi_t, k, h; A), A) \\
given \quad & \psi_1 \\
where \quad & K(\psi_t) = \int_S k \; \psi_t dS, \quad L(\psi_t) = \int_S h \; \psi_t dS
\end{aligned}
$$

Therefore, the planner's optimal policy rules k'^* and $x_h{}^*$ that instruct a household with (k, h, A) today to save $k'^*(k, h; A)$ and invest in human capital $x_h{}^*(k, h; A)$ given the density of today ψ and k'' and h'' (thus, x'_h) tomorrow must maximize

$$
\begin{aligned}
\max_{k', x_h} \quad & \int_S u\left(k[1 + f_K(K(\psi), L(\psi)) - \delta_k] + h f_L(K(\psi), L(\psi)) - k' - x_h\right)\psi dS \\
& + \beta \int_S u\left(k'[1 + f_K(K'(\psi'), L'(\psi')) - \delta_k] + h' f_L(K'(\psi'), L'(\psi')) - k'' - x'_h\right)\psi' dS \\
s.t. \quad & h' = \eta' g(h, x_h, A) \\
where \quad & \psi' \text{ is the density of distribution } \Phi' = T(\Phi, Q(\cdot; k', x_h))
\end{aligned}
$$

which is equivalent to maximizing

$$
\begin{aligned}
\max_{k', x_h} \quad & \int_S u\left(k[1 + f_K(K(\psi), L(\psi)) - \delta_k] + h f_L(K(\psi), L(\psi)) - k' - x_h\right)\psi dS \\
& + \beta \int_S \left[\int_{\eta'} u\left(k'[1 + f_K(K'(\psi'), L'(\psi'))] + h' f_L(K'(\psi'), L'(\psi')) - k'' - x'_h\right) f(\eta')d\eta'\right]\psi dS
\end{aligned}
$$

Then, we derive the first-order conditions variational approach. First, we derive the first-order condition with respect to k'. The variation policy rule,

$$k'^{\epsilon}(\psi,k,h;A) = k'^{*}(\psi,k,h;A) + \epsilon\chi_{h=h_0,A=A_0,k\geq k_0}$$

should be suboptimal. Define

$$\begin{aligned}\Psi_k(\epsilon) &= \int_S [u\left(k[1+f_K(K(\psi),L(\psi))-\delta_k] + hf_L(K(\psi),L(\psi) - k'^{\epsilon}(\psi,k,h,A) - x_h^*(\psi,k,h;A)\right) \\ &\quad +\beta\int_{\eta'} u\left(k'^{\epsilon}(\psi,k,h;A)[1+f_K(K'(T(\psi,k'^{\epsilon})),L'(\psi'))-\delta_k]\right. \\ &\qquad \left. +\eta' g(h,x_h^*,A)f_L(K'(T(\psi,k'^{\epsilon})),L'(\psi')) - k'' - x_h'\right) f(\eta')d\eta'] \psi dS\end{aligned}$$

Then, the derivative with respect to ϵ at 0 of $\Phi_k(\epsilon)$ must be 0.

$$\begin{aligned}\frac{d}{d\epsilon}\Psi_k(0) &= \int_{k_0}^{\infty} [-u'\left(k[1+f_K(K(\psi),L(\psi))-\delta_k] + h_0 f_L(K(\psi),L(\psi))\right. \\ &\qquad \left. -k'^*(\psi,k,h_0;A_0) - x_h^*(\psi,k,h_0;A_0)\right) \\ &\quad +\beta[1+f_K(K'(\psi'),L'(\psi'))-\delta_k]\int_{\eta'} u'\left(k'^*(\psi,k,h_0;A_0)[1+f_K(K'(\psi'),L'(\psi'))-\delta_k]\right. \\ &\qquad \left. +\eta' g(h,x_h^*,A)f_L(K'(\psi'),L'(\psi')) - k'' - x_h'\right) f(\eta')d\eta'] \psi(k,h_0,A_0)dk \\ &+ \beta\int_S\int_{\eta'} f(\eta')\left[\begin{array}{l} u'\left(\begin{array}{l} k'^*(\psi,k,h;A)[1+f_K(K'(\psi'),L'(\psi'))-\delta_k] \\ +\eta' g(h,x_h^*,A)' f_L(K'(\psi'),L'(\psi')) - k'' - x_h' \end{array}\right) \\ \times\{k'^*(\psi,k,h;A)f_{KK}(K'(\psi'),L(\psi')) \\ +\eta' g(h,x_h^*,A)f_{LK}(K'(\psi'),L'(\psi'))\} \times \int_{k_0}^{\infty}\psi(\tilde{k},h_0,A_0)d\tilde{k} \end{array}\right] d\eta'\psi dS \\ &= 0\end{aligned}$$

since the right-hand side of the above equation is a constant function equal to 0. Thus, its derivative with respect to k_0 must be 0. That is, for all k_0, h_0, A_0,

$$\begin{aligned}& -u'\left(k_0[1+f_K(K(\psi),L(\psi))-\delta_k] + h_0 f_L(K(\psi),L(\psi)) - k'^*(\psi,k_0,h_0;A_0) - x_h^*(\psi,k_0,h_0;A_0)\right) \\ &+ \beta[1+f_K(K'(\psi'),L'(\psi'))\delta_k]\int_{\eta'} u'(k'^*(\psi,k_0,h_0;A_0)[1+f_K(K'(\psi'),L'(\psi'))-\delta_k] \\ &\qquad +\eta' g(h_0,x_h^*,A_0)f_L(K'(\psi'),L'(\psi')) - k'' - x_h')f(\eta')d\eta' \\ &+ \beta\int_S\int_{\eta'} f(\eta')[u'\left(k'[1+f_K(K'(\psi'),L'(\psi'))-\delta_k] + \eta' g(h,x_h^*,A)f_L(K'(\psi),L'(\psi)) - k'' - x_h'\right) \\ &\quad \times\left\{k'^*(\psi,k,h;A)f_{KK}(K'(\psi'),L(\psi')) + \eta' g(h,x_h^*,A)f_{LK}(K'(\psi'),L'(\psi'))\right\}]d\eta'\psi dS = 0\end{aligned}$$

That is, for all (k, h, A)

$$
\begin{aligned}
& u'\left(k[1+r(\Phi)]) + hw(\Phi) - k'^{*}(\Phi, k, h; A) - x_h^{*}(\Phi, k, h; A)\right) \\
= \quad & \beta[1+r(\Phi')]E_{\eta'}\left[u'\left(c^{*}(\Phi, \Phi', k, h, \eta'; A)\right)\right] + \Delta_k \\
& where \quad \Delta_k = \beta \int_S u'\left(k'[1+r(\Phi')] + h'w(\Phi) - k'^{*}(\Phi', k', h'; A) - x_h^{*}(\Phi', k', h'; A)\right) \\
& \qquad\qquad \times \left[k' f_{KK}(K', L') - h' F_{LK}(K', L')\right] d\Phi'
\end{aligned}
$$

Next, we derive the first-order condition with respect to x_h. Variation policy rule,

$$
x_h{}^{\epsilon}(\psi, k, h; A) = x_h{}^{*}(\psi, k, h; A) + \epsilon \chi_{k=k_0, A=A_0, h \geq h_0}
$$

should be suboptimal. Define

$$
\begin{aligned}
\Psi_{x_h}(\epsilon) \quad = \quad & \int_S \Big[u\left(k[1+f_K(K(\psi), L(\psi)) - \delta_k] + hf_L(K(\psi), L(\psi) - k'^{*}(\psi, k, h, A) - x_h^{*}(\psi, k, h, A)\right) \\
& + \beta \int_{\eta'} u\Big(k'^{*}(\psi, k, h, A)[1+f_K(K'(\psi'), L'(T(\psi, x_h{}^{\epsilon}))) - \delta_k] \\
& \qquad + \eta' g(h, x_h^{\epsilon}, A) f_L(K'(\psi'), L'(T(\psi, x_h{}^{\epsilon}))) - k'' - x_h'(x_h^{\epsilon}))\Big) f(\eta') d\eta' \Big] \psi dS \\
& where \quad x_h'(x_h{}^{\epsilon}) \text{ is } x_h' \text{ s.t. } \eta'' g(\eta' g(h, x_h{}^{\epsilon}, A), x_h', A) = h''
\end{aligned}
$$

Then, the derivative with respect to ϵ at 0 of $\Psi_{x_h}(\epsilon)$ must be 0.

$$
\begin{aligned}
\frac{d}{d\epsilon}\Psi_{x_h}(0) = & \int_{h_0}^{\infty} \Big[-u'\left(k_0[1+f_K(K(\psi), L(\psi)) - \delta_k] + hf_L(K(\psi), L(\psi)) - k'^{*}(\phi, k_0, h; A_0) - x_h^{*}(\psi, k_0, h, A_0)\right) \\
& + \beta g_{x_h}(h, x_h{}^{*}, A_0) \\
& \quad \times \int_{\eta'} u'(k'^{*}[1+f_K(K'(\psi'), L'(\psi')) - \delta_k] + \eta' g(h, x_h^{*}(\psi, k_0, h; A_0), A_0) f_L(K'(\psi'), L'(\psi')) - k'' - x_h') \\
& \qquad \times \eta' \left\{ f_L(K'(\psi'), L'(\psi')) + \frac{g_h(h', x_h', A_0)}{g_{x_h}(h', x_h', A_0)} \right\} f(\eta') d\eta' \Big] \psi(k_0, h, A_0) dh \\
& + \ \beta \int_S \int_{\eta'} f(\eta') [u'\left(k'[1+f_K(K'(\psi'), L'(\psi'))\delta_k] + h' f_L(K'(\psi'), H'(\psi')) - k'' - x_h'\right) \\
& \quad \times \left\{ k'^{*}(\psi, k, h; A) f_{KL}(K'(\psi'), L'(\psi')) + \eta' g(h, x_h^{*}, A) f_{LL}(K'(\psi'), L(\psi')) \right\} \\
& \quad \times \int_{h_0}^{\infty} g_{x_h}(\tilde{h}, x_h^{*}, A_0) \psi(k_0, \tilde{h}, A_0) d\tilde{h}] d\eta' \psi dS \\
& = \ 0
\end{aligned}
$$

since the right-hand side of the above equation is a constant function equal to 0. Thus, its derivative with respect to h_0 must be 0. That is, for all k_0, h_0, A_0,

$$
\begin{aligned}
& -u'\Big(k_0[1+f_K(K(\psi),L(\psi))-\delta_k]+h_0 f_L(K(\psi),L(\psi))-k'^*(\psi,k_0,h_0;A_0)-x_h{}^*(\psi,k_0,h_0;A_0)\Big) \\
+ \quad & \beta g_{x_h}(h_0,x_h{}^*,A_0)\int_{\eta'} u'(k'^*(\psi,k_0,h_0;A_0)[1+f_K(K'(\psi'),L'(\psi'))-\delta_k] \\
& +\eta' g(h_0,x_h^*,A_0)f_L(K'(\psi'),L'(\psi'))-k''-x_h')\eta'\left\{f_L(K'(\psi'),L'(\psi'))+\frac{g_h(h',x_h',A_0)}{g_{x_h}(h',x_h',A_0)}\right\}f(\eta')d\eta' \\
+ \quad & g_{x_h}(h_0,x_h{}^*,A_0)\beta\int_S\int_{\eta'} f(\eta')[u'\Big(k'^*(\psi,k,h;A)[1+f_K(K'(\psi'),L'(\psi'))-\delta_k] \\
& \quad +\eta' g(h,x_h^*,A)f_L(K'(\psi),L'(\psi))-k''-x_h'\Big) \\
& \times\Big\{k'^*(\psi,k,h;A)f_{KL}(K'(\psi'),L(\psi'))+\eta' g(h,x_h^*,A)f_{LL}(K'(\psi'),L(\psi'))\Big\}]d\eta'\psi dS=0
\end{aligned}
$$

That is, for all (k,h,A)

$$
\begin{aligned}
& u'\Big(k[1+r(\Phi)])+hw(\Phi)-k'^*(\Phi,k,h;A)-x_h{}^*(\Phi,k,h;A)\Big) \\
= \quad & \beta g_{x_h}(h,x_h{}^*(\Phi,k,h;A),A)E_{\eta'}\left[u'(c^*(\Phi,\Phi',k,h,\eta';A))\eta'\left\{w(\Phi')+\frac{g_h(h',x_h',A)}{g_{x_h}(h',x_h',A)}\right\}\right] \\
+ \quad & g_{x_h}(h,x_h{}^*(\Phi,k,h;A),A)\Delta_h \\
& \textit{where} \quad \Delta_h=\beta\int_S\int_{\eta'} f(\eta')\eta'[u'\left(k'[1+f_K(K'(\psi'),L'(\psi'))]+h'f_L(K'(\psi),L'(\psi))-k''-x_h'\right) \\
& \qquad\qquad \times\Big\{k'^*(\psi,k,h;A)f_{KL}(K'(\psi'),L(\psi'))+\eta' g(h,x_h^*,A)f_{LL}(K'(\psi'),L(\psi'))\Big\}]d\eta'\psi dS
\end{aligned}
$$

■

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