Online Appendix:

Screening Adaptive Cartels

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Abstract

This Online Appendix to "Screening Adaptive Cartels" provides extensions and proofs. Section OA establishes a bound on the equilibrium payoff set of the regulatory game. Section OB collects proofs that are not in the main Appendix.

Keywords: collusion, auctions, bidding rings, cartels, procurement, antitrust.

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OA Bounding the Equilibrium Payoff Set

Recall that, for each environment $E \in \mathcal{E}$ and integers T and $M \leq T+1$, $\Sigma_{T,M}(E)$ denotes the set of PPBE of the game with testing date T and monitoring length M. $\Sigma(E)$ denotes the set of PPBE of the game without the regulator.

Fix $E \in \mathcal{E}$, T and $M \leq T+1$. For each strategy profile σ and each history $h_{i,t} = (h_t, z_{i,t})$, let $V_i^E(\sigma, h_{i,t})$ denote player i's expected discounted payoff (excluding possible penalties) at history $h_{i,t}$ under strategy profile σ and environment E. Let $P_i^E(\sigma, h_{i,t}) = \mathbf{1}_{t \leq T} \mathbb{E}_{E,\sigma}\left[\tau_i \delta^{T-t} K_T | h_{i,t}\right]$ denote player i's expected penalty at history $h_{i,t}$ under strategy profile σ and environment E. Player i's total payoff in the regulatory game is $W_i^E(\sigma, h_{i,t}) = V_i^E(\sigma, h_{i,t}) - P_i^E(\sigma, h_{i,t})$.

For T and M, define

$$\mathcal{V}_{T,M}(E) \equiv \left\{ \mathbf{V} \in \mathbb{R}^n : \mathbf{V} = \left(\mathbb{E}_E[W_i^E(\sigma, (h_0, z_i))]\right)_{i \in N} \text{ for some } \sigma \in \Sigma_{T,M}(E) \right\}$$

to be the equilibrium payoff set of the game with testing date T and monitoring length M under environment E. Similarly, let

$$\mathcal{V}(E) \equiv \left\{ \mathbf{V} \in \mathbb{R}^n : \mathbf{V} = \left(\mathbb{E}_E[V_i^E(\sigma, (h_0, z_i))]\right)_{i \in N} \text{ for some } \sigma \in \Sigma(E) \right\}$$

denote the equilibrium payoff set of the game without a regulator.

Let $\overline{B} = B \cup \emptyset$ denote the set of bids and fix $\mathcal{W} \subset \mathbb{R}^n$ a set of possible continuation values. Following Abreu et al. (1990) (henceforth APS), we say that a profile of bidding functions $\beta = (\beta_i)_{i \in N}$, with $\beta_i : z_i \mapsto \beta_i(z_i) \in \Delta(\overline{B})$ for each $i \in N$, is enforceable on $\mathcal{W}$ if there exists $W : \overline{B}^n \to \mathcal{W}$ such that, for all $i \in N$, all z_i, and all $b_i \in \mathsf{supp}\, \beta_i(z_i)$,

$$b_i \in \arg\max_b \mathbb{E}_{E,\beta_{-i}}[(1-\delta)(b - c_i)x_i + \delta W_i(b, \mathbf{b}_{-i}) | z_i],$$

where $x_i \in \{0,1\}$ denotes whether or not i wins the auction.

Fix a profile of bidding functions β enforced by W on $\mathcal{W}$. For each $i \in N$, define

$$v_i^E(\beta, W) \equiv \mathbb{E}_{E,\beta}[(1-\delta)(b_i - c_i)x_i + \delta W_i(b_i, \mathbf{b}_{-i})].$$

For any $\mathcal{W} \subset \mathbb{R}^n$, the APS value-set operator is given by

$$\mathcal{B}_E(\mathcal{W}) \equiv \{\mathbf{V} \in \mathbb{R}^n : \mathbf{V} = (v_i^E(\beta, W))_{i\in N} \text{ for some } \beta \text{ enforced by } W \text{ on } \mathcal{W}\}.$$

Proposition O1 (screening does not increase the value of collusion)**.** *In the game without a regulator, $\mathcal{V}(E) = \lim_{t\to\infty} \mathcal{B}_E^t([0,1]^n)$.*

Consider a regulator running asymptotically safe tests with a false positive rate $G(M)$. Then, for all environments $E \in \mathcal{E}$, $\mathcal{V}_{T,M}(E) \subseteq \mathcal{B}_E^{T-M+2}([-G(M)\delta^{M-1}K_T, 1]^n)$.

Proof. The first statement in the proposition follows from standard arguments (e.g., Abreu et al. (1990)).

For the second statement in the proposition, recall that: (i) for any bidding function β enforced by W, $v_i(\beta, W) = \mathbb{E}_{E,\beta}[(1-\delta)(b_i - c_i)x_i + \delta W_i(b_i, \mathbf{b}_{-i})]$; and (ii) for any set $\mathcal{W} \subset \mathbb{R}^n$,

$$\mathcal{B}_E(\mathcal{W}) = \{\mathbf{V} \in \mathbb{R}^n : \mathbf{V} = (v_i^E(\beta, W))_{i\in N} \text{ for some } \beta \text{ enforced by } W \text{ on } \mathcal{W}\}.$$

The second statement in the proposition then follows from two observations. First, for every strategy profile $\overline{\sigma}_{-i}$ of i's opponents and every history $h_{i,t}$ with $t = T - M + 1$, firm i can guarantee itself a payoff of at least $0 - G(M)\delta^{M-1}K_T$ by playing a static best-response to $\overline{\sigma}_{-i}$ at all periods $s \geq t$. Second, since firms' flow profits are bounded above by $r = 1$, we have that $W^E(\overline{\sigma}, h_{i,t}) \leq 1$ for all i, all $\overline{\sigma}$ and all $h_{i,t}$. Hence, for any $E \in \mathcal{E}$ and any $\sigma \in \Sigma_{T,M}(E)$, firms' payoffs (including expected penalties) at $t < T - M + 1$ must lie in $\mathcal{B}_E^{T-M+1-t}([-G(M)\delta^{M-1}K_T, 1]^n)$. ■

In words, we can bound the equilibrium payoff set of the regulatory game by applying the APS operator $T - M + 2$ times to the set $[-G(M)\delta^{M-1}K_T, 1]^n$. If the limit $\cap_{t\geq 0}\mathcal{B}_E^t([-\nu, 1]^n)$ depends continuously on ν, and T, M are chosen so that $G(M)\delta^{M-1}K_T$ becomes vanishingly small as M and T grow large, then the set of values under screening $\mathcal{V}_{T,M}(E)$ will be included in the set of values without a regulator $\mathcal{V}(E)$ as $T - M$ and M grow large.[1]

The key intuition behind Proposition O1 is that screening for collusion using safe tests does not significantly lower firms' min-max values. At any period $t \leq T - M + 1$, a firm can guarantee to pass the test with probability $1 - G(M)$ by playing a static best response to its rivals during the monitoring phase. Hence, since firms' continuation payoffs at $t = T - M + 2$ must lie in $[-G(M)\delta^{M-1}K_T, 1]^n$, we can bound firms' equilibrium payoffs at the start of the game by applying the APS operator $T - M + 2$ times (the number of periods between 0 and $T - M + 1$) to $[-G(M)\delta^{M-1}K_T, 1]^n$.[2]

OB Additional Proofs

Proof of Corollary 1. Fix $E \in \mathcal{E}$, $\sigma \in \Sigma_{T,M}(E)$, $i \in N$ and a history $h_{i,T-M+1}$. Firm i's expected payoff at $h_{i,T-M+1}$ under σ, E is $V_i^E(\sigma, h_{i,T-M+1}) - P_i^E(\sigma, h_{i,T-M+1})$. Let $\hat{\sigma}_i$ be a strategy under which firm i plays a static best-response to σ_{-i} at all histories. Firm i's payoff at history h_{T-M+1} from playing according to $\hat{\sigma}_i$ when the other firms play according to σ_{-i} satisfies

$$V_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M+1}) - P_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M+1}) \geq 0 - G(M)\delta^{M-1}K_T = -G(M)\delta^{-(T-M+1)}K, \tag{O1}$$

where the inequality follows since τ_i is an asymptotically safe test and since firm i's static best response each period must give i a payoff weakly larger than 0, and the equality uses

[1] We note that, in general, $\cap_{t\geq 0}\mathcal{B}_E^t([-\nu, 1]^n)$ may not be continuous in ν.

[2] We note that the same would be true if safe tests (τ_i) depended on the entire bidding data up to T, but gave a growing weight to data from the last M periods, so that a firm that behaves competitively in the last M periods would be guaranteed to pass with high probability.

$K_T = \delta^{-T} K$. Since σ is an equilibrium, and since $V_i^E(\sigma, h_{i,T-M+1}) \leq r = 1$, we have

$$
\begin{aligned}
1 - P_i^E(\sigma, h_{i,T-M+1}) &\geq V_i^E(\sigma, h_{i,T-M+1}) - P_i^E(\sigma, h_{i,T-M+1}) \\
&\geq V_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M+1}) - P_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M+1}) \geq -G(M)\delta^{-(T-M+1)}K \\
&\implies \quad P_i^E(\sigma, h_{i,T-M+1}) \leq 1 + G(M)\delta^{-(T-M+1)}K.
\end{aligned}
$$

Using $P_i^E(\sigma, h_{i,T-M+1}) = \delta^{M-1} K_T \mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M+1}\right] = \delta^{-(T-M+1)} K \mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M+1}\right]$, we get

$$
\mathsf{prob}_{E,\sigma}(\tau_i = 1 | h_{i,T-M+1}) = \mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M+1}\right] \leq \frac{1}{\delta^{-(T-M+1)}K} + G(M).
$$

Hence, if $M = \lfloor T^x \rfloor$ for some $x \in (0,1)$, $\mathsf{prob}_{E,\sigma}(\tau_i = 1 | h_{i,T-M+1})$ converges to 0 as $T \to \infty$. This completes the proof. ■

Proof of Proposition 4. Consider first part (i). Since the auction runs only if both firms participate, the cartel's payoff in the repeated game is bounded above by $(r - 2\kappa) = (1 - 2\kappa)$. Consider the following strategy σ_i^{coll}:

- at the initial history h_0, or at any history h_t with $b_{j,s} = r$ for $j = 1, 2$, $s < t$, bid $b_{i,t} = r$;
- at any other history h_t, play the static Nash equilibrium.[3]

One can verify that, when $\delta \geq \underline{\delta} = \frac{1}{1-\kappa}\frac{1}{2}$, both firms playing according to σ_i^{coll} is an equilibrium of the repeated game without a regulator. Hence, $\overline{V}^{\mathsf{bmk}} = 1 - 2\kappa$.

We now turn to part (ii). We start by providing an upper bound to the cartel's payoffs during the monitoring phase. Let $b_{(1),s}$ denote the winning bid at period s, and for any $t \in [T - M + 1, T + 1]$ let $B_{t-1} = \sum_{s=T-M+1}^{t-1} b_{(1),s}$ denote the cumulative sum of winning bids during the testing phase prior to t (with $B_{T-M} = 0$). Note that, if both firms participate at

[3] Under our assumption that the auction doesn't run with only one participant, the stage game admits a Nash equilibrium in which both firms stay out of the auction.

all periods $t = T - M + 1, ..., T$, at least one firm $i \in \{1, 2\}$ fails test τ_i^{close} if $\frac{1}{M} B_T > \overline{B} \equiv 1 - (1 - 2\rho)\Delta$. Define $\widehat{W} \equiv \overline{B} - 2\kappa$.

For each history $h_t = (\mathbf{b}_s)_{s<t}$ with $t \in [T - M + 1, T + 1]$ define the state $z_t = (t, B_{t-1})$ and let

$$\Phi(z_{T+1}) = 1 - 2\kappa - \delta^{-1} K_T \mathbf{1}_{B_T > M \times \overline{B}}.$$

Note that $\delta\Phi(z_{T+1})$ is an upper bound on the sum of firms' equilibrium continuation payoffs at the end of period T, after bidding is done but before any penalties are paid.

For each $z_t = (t, B_{t-1})$ with $t \in [T + 1 - M, T]$, define

$$\begin{aligned} \Phi(z_t) = \sup_{b_t \sim F \in \Delta([0,1])} & \mathbb{E}_F[(1 - \delta)(b_t - 2\kappa) + \delta\Phi(t + 1, B_{t-1} + b_t)] \text{ s.t.} \\ \forall b_t \in \mathsf{supp}\, F, \quad & b_t \leq \frac{\delta}{1 - \delta}\Phi(t + 1, B_{t-1} + b_t), \end{aligned} \tag{O2}$$

with $\Phi(z_{T+1}) = 1 - 2\kappa - \delta^{-1} K_T \mathbf{1}_{B_T > M \times \overline{B}}$. Two points are worth noting. First, (O2) always admits a deterministic solution (i.e., a solution such that winning bid b_t is deterministic for all z_t). Indeed, (O2) is a single agent dynamic programing problem, and hence admits a deterministic solution. Second, whenever the penalty K_T is sufficiently large (e.g., larger than $\overline{K} \equiv \delta^{-T}(1 - 2\kappa)$), the solution to (O2) will be such that firms pass the test with probability 1: i.e., $B_T \leq M\overline{B}$. From now on, we assume $K_T \geq \overline{K}$. For each $t = T + 1 - M, ..., T + 1$, let $\overline{\Phi}_t$ be the value of Program (O2) at t. Since $z_{T+1-M} = (T + 1 - M, 0)$ regardless of the bidding history, the solution to (O2) does not depend on bidding behavior prior to $T + 1 - M$.

We now show that, for any equilibrium σ and any history h_{T+1-M} of length $T + 1 - M$, the cartel's payoff under σ at history h_{T+1-M} is bounded above by $\overline{\Phi}_{T+1-M}$. To establish this result, we show that for any period t during the testing phase with history $h_t = (\mathbf{b}_s)_{s<t}$ and with state $z_t = (t, B_{t-1})$, if history h_t is such that each firm $i \in \{1, 2\}$ would pass the test if it were to not participate at any auction $s \geq t$, then the cartel's equilibrium payoff at h_t under σ is bounded above by $\Phi(z_t)$. Note that this implies that $\overline{\Phi}_{T+M-1}$ is an upper

bound to equilibrium payoffs at any history h_{T+1-M}.[4]

Note first that, since $\Phi(z_{T+1}) = 1 - 2\kappa$ for any history h_{T+1} such that both firms pass the test, cartel equilibrium payoffs at such a history h_{T+1} are bounded above by $\Phi(z_{T+1})$. Towards an induction, suppose that the result holds for all histories $h_{\hat{s}}$ of length $\hat{s} = t + 1, ..., T + 1$ with the property that each firm would pass the test if it stopped participating at all $s \geq \hat{s}$. Fix an equilibrium σ and a history $h_t = (\mathbf{b}_s)_{s<t}$ of length t, with t during the monitoring phase, with the property that each firm would pass the test if it stopped participating. Let $b_{(1),t}$ denote the equilibrium winning bid at h_t. Note that, if both firms participate at h_t, then for $i = 1, 2$ we must have

$$(1 - \delta)(x_{i,t} b_{(1),t} - \kappa) + \delta W_{i,t+1} \geq (1 - \delta)(b_{(1),t} - \kappa), \tag{O3}$$

where $x_{i,t}$ denotes the probability with which i wins the auction at time t and $W_{i,t+1}$ denotes i's continuation value given history $h_{t+1} = h_t \sqcup \mathbf{b}_t$ (including expected penalties).[5] Summing across both players, and using $x_{1,t} + x_{2,t} \leq 1$ and $W_{1,t+1} + W_{2,t+1} = W_{t+1}$, we get

$$\begin{gathered}(1 - \delta)(b_{(1),t} - 2\kappa) + \delta W_{t+1} \geq 2(1 - \delta)(b_{(1),t} - \kappa) \\ \iff \frac{\delta}{1 - \delta} W_{t+1} \geq b_{(1),t}.\end{gathered}$$

If bids $\mathbf{b}_t = (b_{1,t}, b_{2,t})$ are such that firm $i \in \{1, 2\}$ fails the test at history $h_t \sqcup \mathbf{b}_t$ if it were to stop participating, then we have $W_t \leq \Phi(z_t)$ whenever $K \geq \overline{K} = \delta^{-T}(1 - 2\kappa)$. If not, then by the induction hypothesis we have that $W_{t+1} \leq \Phi(z_{t+1})$; and so it must be that

$$W_t = (1 - \delta)(b_{(1),t} - 2\kappa) + \delta W_{t+1} \leq \Phi(z_t),$$

[4] Indeed, all histories h_{T+1-M} of length $T+1-M$ have $B_{T-M} = 0$ and have the property that each firm would pass test τ_i^{close} if it didn't participate at any auction $s \geq T + 1 - M$.

[5] To see why (O3) must hold under σ, recall that h_t is such that i would pass the test if it were to not participate any longer. Hence, (O3) must hold since firm i can obtain a payoff equal to the right-hand side by undercutting bid $b_{(1),t}$ at t and not participating at any future date.

where the inequality follows since $\frac{\delta}{1-\delta}\Phi(z_{t+1}) \geq \frac{\delta}{1-\delta}W_{t+1} \geq b_{(1),t}$. Hence, equilibrium payoffs at any history h_{T-M+1} are bounded above by $\overline{\Phi}_{T-M+1}$.

We now show that, for all $\epsilon > 0$, $\overline{\Phi}_t \leq \widehat{W} + \epsilon = \overline{B} - 2\kappa + \epsilon$ for some $t \in [T-M+1, T]$ whenever M is large enough. Since the solution to (O2) is deterministic, we have that for all $t = T-M+1, ..., T$, $\overline{\Phi}_t = (1-\delta)(b_t - 2\kappa) + \delta\overline{\Phi}_{t+1}$, or

$$b_t = \frac{1}{1-\delta}\left(\overline{\Phi}_t - \delta\overline{\Phi}_{t+1}\right) + 2\kappa.$$

Summing over periods $t = T-M+1, ..., T$ and dividing by M we get

$$\frac{1}{M}\sum_{t=T-M+1}^{T} b_t = \frac{1}{M}B_T = \frac{1}{M}\left(\sum_{t=T-M+2}^{T}\overline{\Phi}_t + \frac{1}{1-\delta}\left(\overline{\Phi}_{T-M+1} - \delta\overline{\Phi}_{T+1}\right)\right) + 2\kappa. \tag{O4}$$

Since firms pass the test under the solution to (O2), we have $\frac{1}{M}B_T \leq \overline{B} = \widehat{W} + 2\kappa$, and so (O4) gives us

$$\frac{1}{M-1}\sum_{t=T-M+2}^{T}\overline{\Phi}_t \leq \frac{M}{M-1}\widehat{W} + \frac{1}{1-\delta}\frac{1}{M-1}\left(\delta\overline{\Phi}_{T+1} - \overline{\Phi}_{T-M+1}\right)$$

Since $\overline{\Phi}_t \leq 1 - 2\kappa$ for all t, we have that for all $\epsilon > 0$ there exists $\overline{M}_\epsilon$ such that, for all $M \geq \overline{M}_\epsilon$,

$$\frac{1}{M-1}\sum_{t=T-M+2}^{T}\overline{\Phi}_t \leq \widehat{W} + \epsilon. \tag{O5}$$

Equation (O5) implies that there exists $t \in [T-M+2, T]$ such that $\overline{\Phi}_t \leq \widehat{W} + \epsilon$ whenever $M \geq M_\epsilon$.

Pick $\epsilon > 0$ such that $\widehat{W} + \epsilon < 1 - 2\kappa$. Note that such an $\epsilon > 0$ exists since $\overline{B} \equiv 1 - (1-2\rho)\Delta$ and $\widehat{W} \equiv \overline{B} - 2\kappa$. Assume $M \geq \overline{M} \equiv \overline{M}_\epsilon$. Note then that there exists $\mu > 0$ and $\eta > 0$ such that, for all $V \leq \widehat{W} + \epsilon$, $\frac{\delta}{1-\delta} V \leq V + 2\kappa - \eta$ for all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$.[6] We now show that, for all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$, $\overline{\Phi}_{T-M+1} \leq \widehat{W} + \epsilon$. From our arguments above, we know that there exists $t \in [T - M + 2, T]$ such that $\overline{\Phi}_t \leq \widehat{W} + \epsilon$. Note then that

$$\begin{aligned} \overline{\Phi}_{t-1} &= (1-\delta)(b_{t-1} - 2\kappa) + \delta \overline{\Phi}_t \\ &\leq (1-\delta)\left(\frac{\delta}{1-\delta} \overline{\Phi}_t - 2\kappa \right) + \delta \overline{\Phi}_t \\ &\leq \widehat{W} + \epsilon - (1-\delta)\eta, \end{aligned}$$

where the first inequality uses $b_{t-1} \leq \frac{\delta}{1-\delta} \overline{\Phi}_t$, and the second inequality uses $\overline{\Phi}_t \leq \widehat{W} + \epsilon$ and $\frac{\delta}{1-\delta} \overline{\Phi}_t \leq \overline{\Phi}_t + 2\kappa - \eta$. If $t - 1 = T - M + 1$, we are done. Otherwise, we can repeat the same argument until we obtain $\overline{\Phi}_{T-M+1} < \widehat{W} + \epsilon$.

Finally, we show that for all $\varepsilon > 0$ and all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$, there exists $\overline{T}$ such that $\overline{V}^{\mathsf{bmk}}_{T,M} < \varepsilon$ for all $M > \overline{M}$, $T - M > \overline{T}$.

Fix $\sigma \in \Sigma_{T,M}(E)$ and a history h_t with $t \leq T - M$. Let $b_{(1),t}$ denote the winning bid at h_t. Note that, if both firms participate, then for $i = 1, 2$ we must have

$$(1-\delta)(x_{i,t} b_{(1),t} - \kappa) + \delta W_{i,t+1} \geq (1-\delta)(b_{(1),t} - \kappa),$$

where $x_{i,t}$ denotes the probability with which i wins the auction at time t and $W_{i,t+1}$ denotes i's continuation value given history $h_{t+1} = h_t \sqcup \mathbf{b}_t$ (including possible penalties). Summing

[6] To see why, note that, for all $V \leq \widehat{W} + \epsilon$, we have

$$V + 2\kappa - \frac{\underline{\delta}}{1-\underline{\delta}} V = 2\kappa \left(1 - \frac{V}{1-2\kappa} \right) \geq 2\kappa \left(1 - \frac{\widehat{W} + \epsilon}{1 - 2\kappa} \right) > 0,$$

where the equality follows since $\frac{\underline{\delta}}{1-\underline{\delta}} = \frac{1}{1-2\kappa}$, and the strict inequality follows since $\widehat{W} + \epsilon < 1 - 2\kappa$. Hence, for $\eta > 0$ and $\mu > 0$ small enough, we have that $\frac{\delta}{1-\delta} V \leq V + 2\kappa - \eta$ for all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$.

across both players, and using $x_{1,t} + x_{2,t} \leq 1$ and $W_{1,t+1} + W_{2,t+1} = W_{t+1}$, we get

$$\begin{aligned} &(1-\delta)(b_{(1),t} - 2\kappa) + \delta W_{t+1} \geq 2(1-\delta)(b_{(1),t} - \kappa) \\ &\iff \frac{\delta}{1-\delta} W_{t+1} \geq b_{(1),t}. \end{aligned}$$

For each value $W \in \mathbb{R}_+$, define operator $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\Psi(W) \equiv (1-\delta)\left(\max\left\{\min\left\{1, \frac{\delta}{1-\delta}W\right\} - 2\kappa, 0\right\}\right) + \delta W.$$

Then, for any history h_t with $t \leq T - M$, $\Psi(W)$ is an upper bound to the cartel's payoff at a history h_t whenever the cartel's continuation value at $t+1$ is bounded above by W. Since by our arguments above cartel's continuation payoff at any history h_{T-M+1} is bounded above by $\overline{\Phi}_{T-M+1} \leq \widehat{W} + \epsilon$, we have that $\overline{V}^{\mathsf{bmk}}_{T,M} \leq \Psi^{T-M}(\widehat{W} + \epsilon)$. Recall that $\frac{\delta}{1-\delta} W \leq W + 2\kappa - \eta$ for all $W \leq \widehat{W} + \epsilon$. Hence, for all $W \leq \widehat{W} + \epsilon$ we have $\Psi(W) \leq \max\{W - (1-\delta)\eta, \delta W\}$. And so, for $T - M$ sufficiently large, $\varepsilon > \Psi^{T-M}(\widehat{W} + \epsilon) \geq \overline{V}^{\mathsf{bmk}}_{T,M}$. This completes the proof.

■

References

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