Supplementary Appendix
"Data Linkages and Privacy Regulation"

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C Nonlinear Equilibria

Weng, Wu and Yin (2023) study nonlinear separating Perfect Bayesian Equilibria in linear-quadratic games such as ours. For the special case $\varphi = 1,\ k = 0,\ c = 1, \sigma_2 = 0$, i.e. perfectly persistent types, they show that our game admits a unique nonlinear PBE. In particular, let the consumer's type $\theta \in [\underline{\theta}, \bar{\theta}]$ and consider the consumer's first-period PBE strategy $q_1(\theta, p_1, y_1)$.

When $\lambda_2 < 0$, Weng, Wu and Yin (2023, Section 5.2) show that this strategy takes the form

$$\frac{(q_1 - (\theta + b_1 y_1 - p_1) - \theta\lambda_2)^{\lambda_2}}{[q_1 - (\theta + b_1 y_1 - p_1) + (1+\lambda_2)\theta]^{-1-\lambda_2}} = \frac{(-\bar{\theta}\lambda_2)^{\lambda_2}}{[(1+\lambda_2)\bar{\theta}]^{-1-\lambda_2}}. \tag{46}$$

When $\lambda_2 > 0$, this strategy takes the following form:

$$\frac{(-q_1 + (\theta + b_1 y_1 - p_1) + \theta\lambda_2)^{\lambda_2}}{[q_1 - (\theta + b_1 y_1 - p_1) + (1+\lambda_2)\theta]^{-1-\lambda_2}} = \frac{(\underline{\theta}\lambda_2)^{\lambda_2}}{[(1+\lambda_2)\underline{\theta}]^{-1-\lambda_2}}. \tag{47}$$

Under these strategies, the highest (resp. lowest) consumer type chooses the myopic strategy $q_1 = \theta + b_1 y_1 - p_1$ when $\lambda_2 < 0$ (resp. > 0). Figure 10 compares the Weng, Wu and Yin (2023) with the myopic consumer's best reply to prices and quality p_1, y_1 and with the linear equilibrium strategy in our paper.

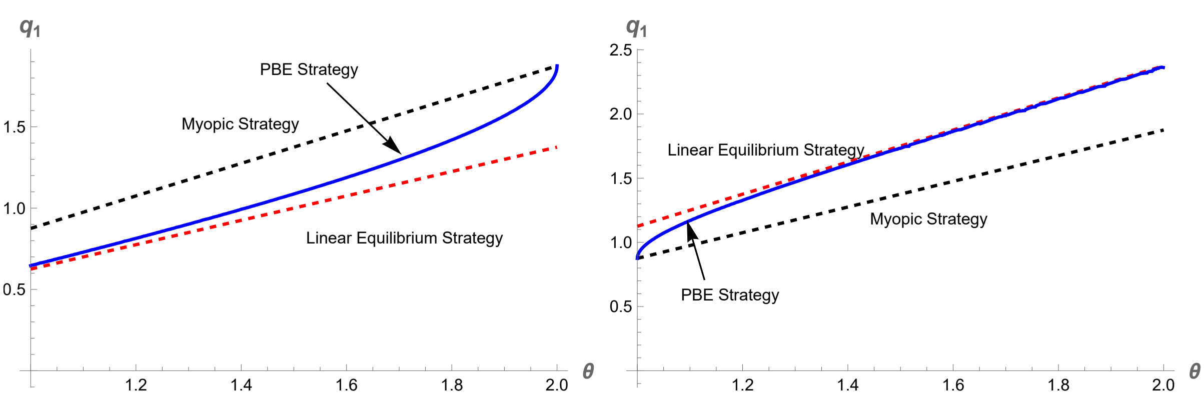

Figure 10: Linear and Perfect Bayesian Equilibria (fixed support)

The characterization in Weng, Wu and Yin (2023) implies (but their paper does not show) that, for all $\theta < \bar{\theta}$, the Perfect Bayesian Equilibrium strategy converges pointwise to our Linear Bayesian equilibrium strategy as the support of the type distribution grows. In particular, let q^{LIN} and q^{PBE} denote the Linear and Perfect Bayesian equilibrium strategies, respectively. We then obtain the following new result.

Proposition 11 (Approximation). *If $\lambda_2 < 0$, we have $q^{PBE}(\theta) \to q^{LIN}(\theta)$ for every $\theta < \bar{\theta}$ as $\bar{\theta} \to \infty$. If $\lambda_2 > 0$, we have $q^{PBE}(\theta) \to q^{LIN}(\theta)$ for every θ as $\underline{\theta} \to 0$.*

Proof of Proposition 11. Consider the case $\lambda_2 < 0$ first. The closed-form solution for the PBE strategy is provided in equation (46). Fix θ and let $\bar{\theta} \to \infty$. Because $1 + 2\lambda_2 > 0$, the right-hand side of (46) diverges, but the denominator on the left-hand side remains bounded because the solution to the best-response problem of each type $\theta < \bar{\theta}$ is given by a finite q_1. Therefore, the numerator must vanish (because the exponent is $\lambda_2 < 0$), which occurs at the linear equilibrium strategy

$$q_1 = (1 + \lambda_2)\theta + b_1 y_1 - p_1.$$

For the case of $\lambda_2 > 0$, Weng, Wu and Yin (2023) establish that the PBE and linear equilibrium coincide when $\underline{\theta} = 0$. The convergence result then follows from the continuity of (47) in q and $\underline{\theta}$ for every θ. $\square$

In Figure 11, we augment the picture in the Weng, Wu and Yin (2023) paper by comparing the PBE strategies for different supports of the consumer's type. (Note that the PBE first-period price and quality p_1 and y_1 depend on the distribution of types, and as such, they will vary with the support. In Figure 11, we hold them fixed to highlight the consumer's equilibrium strategy $q_1(\theta_1)$.)

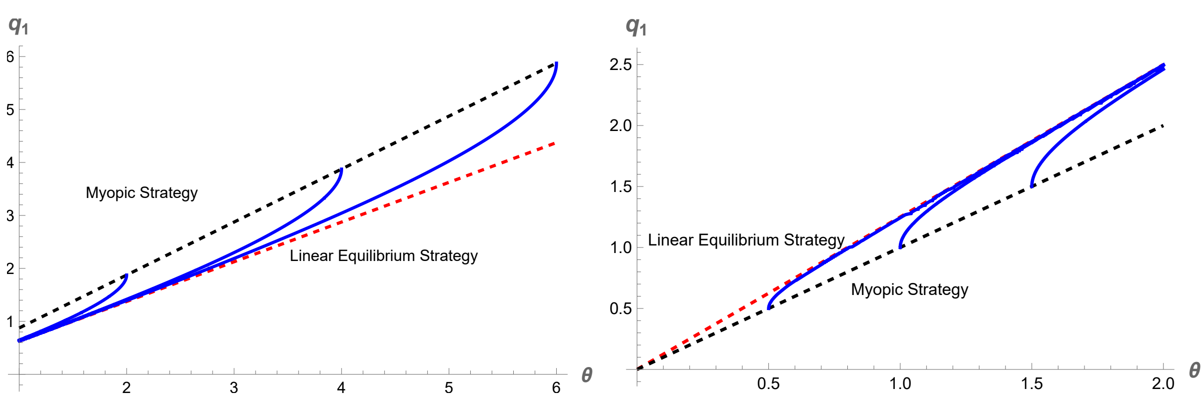

Figure 11: Linear and Perfect Bayesian Equilibria (varying support)

Weng, Xi, Fan Wu, and Xundong Yin. 2023. Linear Riley equilibria in quadratic signaling games. *Journal of Economic Theory*, 213.

D Two-Part Tariffs

We now relax the assumption that the second-period firm must adopt a linear pricing rule for each consumer. Specifically, we now assume that the second-period firm is able to offer a single two-part tariff $T + pq$ that conditions on the first-period information.

In this section, we derive conditions under which a linear equilibrium exists in both periods. In such an equilibrium, the second-period firm's quality provision, as well as the fixed and variable parts of the optimal tariff are linear in the first period outcome data. Furthermore, the consumer's quantity choice is a linear function of the current-period terms of trade. In particular, the consumer's strategy in the first period is given by

$$q_1 = \alpha\theta_1 + \beta y_1 + \gamma p_1 + \delta. \tag{48}$$

Throughout this section, we let $\varepsilon_2 \in [-\Delta, \Delta]$ and assume Δ is small enough that the second-period firm offers a two-part tariff that serves every type θ_2. We then obtain the following result.

Proposition 12 (Linear Equilibrium)**.** *If* $\lambda_2 \in (-4\varphi^2/(1+4\varphi^2), 0)$*, there exists a unique linear equilibrium in which the consumer plays strategy* (48) *with*

$$\alpha^* = \sqrt{\frac{(\lambda_2+1)\varphi^2}{\lambda_2} + \frac{1}{4}} + \frac{1}{2}, \quad \beta^* = b_1, \quad \gamma^* = -1, \quad \delta^* = \frac{(1-\alpha^*)(2\Delta + \mu(\varphi - 1))}{\varphi}.$$

Proof. Consider the second period. In any separating (e.g., linear) equilibrium, firm 2 holds degenerate beliefs over the consumer's first-period type θ_1. Under our full coverage assumption (i.e., Δ small), the fixed part of the optimal two-part tariff for firm 2 extracts the entire willingness to pay of the lowest type in the support of its beliefs. Because the consumer's second-period demand function is given by

$$q_2 = \theta_2 + b_2 y - p,$$

the fixed part of the optimal second-period tariff is given by

$$T^*(m) = \frac{(m - \Delta + b_2 y - p)^2}{2}.$$

Consequently, the second-period firm maximizes the following profit function

$$\Pi_2(p, y) := \left[(p - ky)(m + b_2 y - p) - c_2 y^2/2 + (m - \Delta + b_2 y - p)^2/2\right].$$

Note that the function Π_2 is globally concave if and only if

$$\lambda_2 := \frac{(b_2 - k)^2 - c_2}{2c_2 - (b_2 - k)^2} < 0.$$

When $\lambda_2 < 0$ the second-period firm's optimal (variable) price and quality satisfy

$$p_2^*(m) = \frac{\Delta\left(\lambda_2(b_2 + k) + k\right) - k(2\lambda_2 + 1)m}{\lambda_2(b_2 - k)}, \quad y_2^*(m) = \frac{(2\lambda_2 + 1)(\Delta - m)}{\lambda_2(b_2 - k)}.$$

These price and quality levels yield the following indirect utility function for the consumer:

$$U_2(\theta, m) = \frac{\left(\Delta - m + \theta_2\right)\left(2\Delta - 2m + (\Delta - 3m + \theta_2)\lambda_2\right)}{2\lambda_2}.$$

Under the linear conjecture for the first-period demand function by the consumer (26) and upon observing the outcome of the first period, firm 2 forms (degenerate) beliefs about θ_1,

$$m_2(q_1, p_1, y_1) = \varphi \frac{q1 - (\beta y_1 + \gamma p_1 + \delta)}{\alpha} + (1 - \varphi)\mu.$$

Under this conjecture, the consumer then solves the following problem in the first period

$$q_1 = \arg\max_q \left[q_1(\theta_1 + b_1 y_1 - p_1) - q_1^2/2 + U_2(\varphi\theta_1 + (1 - \varphi)\mu, m(q_1, p_1, y_1))\right]. \tag{49}$$

Taking the first-order condition in problem (49) and matching coefficients with the conjectured strategy (48) yields the result. □

In particular, Proposition 12 implies that $\alpha^* < 1$ for all $\lambda_2 < 0$ that admit a linear equilibrium, as in the baseline analysis (Proposition 2). The major difference with our baseline setting is that for $\lambda_2 > 0$ or φ large, the returns to quality are unbounded and hence no linear equilibrium exists in the static game.

However, whenever a linear equilibrium exists, it has qualitatively similar implications for markets for information as under linear pricing. Figure 12 shows the set of (λ_1, λ_2) pairs for which total producer surplus is higher with a data linkage than without.

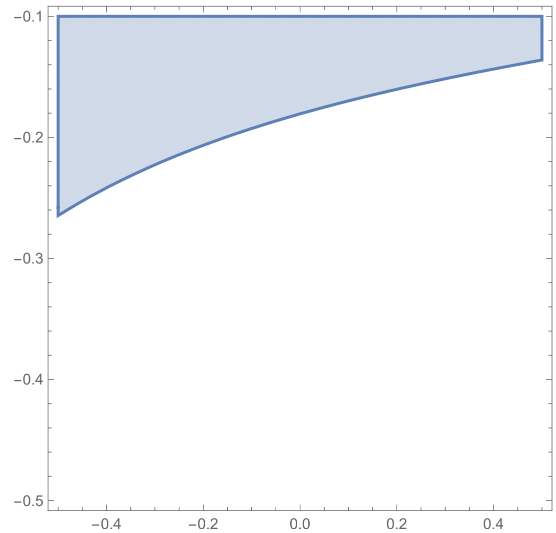

Figure 12: Linkages $(\lambda_1, \lambda_2) \in \Lambda^{PS}$ with two-part tariffs

Absent regulation, the pairs of firms in the shaded region will actively trade the consumer's information. The parameter values are $\mu = 1, \varphi = 1/6$, and the first and second period types ε_t are uniformly distributed on $[-4/9, 4/9]$. (The details of the calculations of producer surplus are available from the authors.)

E Noisy Data

We now extend our baseline model to the case where the second-period firm observes the first-period outcome with noise. For ease of exposition, we set the marginal cost of production $k = 0$, and we let types be fully persistent, i.e., $\varphi = 1$. We then introduce the following information structure.

The consumer's first-period type $\theta \in \mathbb{R}$ is drawn from a Gaussian distribution

$$\theta \sim N(\mu, 1/\tau_\theta).$$

Under a data linkage, the second-period firm observes the realized p_1, y_1 as well as a signal of the consumer's interaction with firm 1,

$$s \sim N(q_1, 1/\tau_q).$$

The analysis in the second period is unchanged relative to the baseline model—the consumer's second-period payoff can be written as a function of her type and the firm's posterior mean beliefs m,

$$U_2(\theta, m) = \frac{1}{2}(\theta + \lambda_2 m)^2, \quad \text{where } \lambda_2 \triangleq \frac{b_2^2 - c_2}{2c_2 - b_2^2}.$$

In the first period, the consumer chooses q_1 to solve the following problem,

$$U_1(\theta, p_1, y_1) = \max_q \left[(\theta + b_1 y_1 - p_1)q - q^2/2 + \int U_2(\theta, m)\, \mathrm{d}F(m \mid q)\right],$$

where $F(m \mid q)$ denotes the distribution of firm 2's posterior mean m given the consumer's quantity choice and the firm's conjecture about the consumer's strategy. Under the Gaussian information structure we have assumed, the consumer's choice of q leads to shifts in $F(m)$. It is then natural to look for a linear equilibrium.

Proposition 13 (Linear Equilibrium with Noisy Signal)**.** *There exists a unique equilibrium in linear strategies. In this equilibrium, the consumer's first period demand is given by*

$$q_1^*(\theta, p_1, y_1) = \alpha^*\theta + b_1 y_1 - p_1 + \delta^*,$$

where α^* solves

$$\lambda_2 = (\alpha^* - 1)\left(\frac{\tau_\theta}{\alpha^{*2}\tau_q} + 1\right), \tag{50}$$

and

$$\delta^* = \frac{\alpha^* \lambda_2^2 \mu \tau_q \tau_\theta}{\left(\alpha^{*2}\tau_q + \tau_\theta\right)^2}.$$

Proof of Proposition 13. To characterize a linear equilibrium, suppose firm 2 conjectures a linear strategy

$$\hat{q}_1\left(\theta, p_1, y_1\right) = \hat{\alpha}\theta + \hat{\beta}y_1 + \hat{\gamma}p_1 + \hat{\delta}. \tag{51}$$

The consumer's first-period best reply is then given by

$$q_1\left(\theta, p_1, y_1\right) = \theta + b_1 y_1 - p_1 + \lambda_2 \frac{\partial \mathbb{E}\left[m \mid q\right]}{\partial q}\left(\theta + \lambda_2 \mathbb{E}\left[m \mid q\right]\right).$$

Under a conjectured linear strategy (51) for the consumer, the second-period firm's posterior mean as a function of the realized signal s is given by

$$m := \mathbb{E}\left[\theta \mid s\right] = \frac{\mu\tau_\theta + \frac{s - \hat{\delta} - \hat{\gamma}p_1 - \hat{\beta}y_1}{\hat{\alpha}}\hat{\alpha}^2\tau_q}{\tau_\theta + \hat{\alpha}^2\tau_q}.$$

From the consumer's perspective, the expectation of the firm's belief m is a function of the first-period quantity choice, which is given by

$$\mathbb{E}\left[m \mid q\right] = \frac{\mu\tau_\theta + \frac{q - \hat{\delta} - \hat{\gamma}p_1 - \hat{\beta}y_1}{\hat{\alpha}}\hat{\alpha}^2\tau_q}{\tau_\theta + \hat{\alpha}^2\tau_q}.$$

Substituting in the linear best reply for the consumer and matching the coefficients yields the result. □

The linear equilibrium of the model with noisy observations has the following properties, which are immediate from condition (50).

Corollary 2 (Equilibrium Properties)**.**

1. *α^* is continuous and strictly increasing in λ_2, with $\alpha^* = 1$ if $\lambda_2 = 0$.*
2. *$\alpha^* > 0$ is decreasing in the signal precision τ_q if $\lambda_2 < 0$ and increasing if $\lambda_2 > 0$.*
3. *For all $\lambda_2 > -1/2$, $\alpha^* \to 1 + \lambda_2$ as $\tau_q \to \infty$.*

Therefore, as expected, the linear equilibrium of a model with noisy observations converges to the linear equilibrium of the baseline model as the precision of the first-period quantity signal grows without bound.

F Direct Payments for Consent

We now describe the equilibrium outcome under a complete and efficient market for consumer information. In this section, we assume that transparency and consumer consent are required and that firm 1 is allowed to offer a direct (positive or negative) payment to the consumer in exchange for her consent to form a linkage with firm 2. For ease of exposition only, we let $\varphi = 1$ and $\varepsilon_2 = 0$, and we assume that firm 1 has all the bargaining power vis-à-vis firm 2 and the consumer. That is, firm 1 extracts all the surplus from the formation of a link. As the two firms bargain efficiently, firm 1 proposes forming the linkage (λ_1, λ_2) if and only if this linkage increases *social* surplus. Proposition 14 establishes our characterization result.

Proposition 14 (Social Welfare)**.**
There exist two thresholds $\tilde{\lambda}_2(\lambda_1, \tilde{\sigma})$ and $\tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma})$ satisfying $\tilde{\lambda}_2(\lambda_1, \tilde{\sigma}) < 0$ for all $\lambda_1, \tilde{\sigma} \geq 0$, and $\tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma}) > 0$ for all $\lambda_1 < 0 < \tilde{\sigma}$, such that the following hold.

1. *For $\lambda_1 \geq 0$ all linkages with $\lambda_2 \geq \tilde{\lambda}_2(\lambda_1, \tilde{\sigma})$ increase social welfare.*
2. *For $\lambda_1 < 0$, all linkages $\lambda_2 \in [\tilde{\lambda}_2(\lambda_1, \tilde{\sigma}), \tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma})]$ increase social welfare.*

Proof of Proposition 14. The total change in social welfare ΔW due to the formation of a linkage can be obtained by adding ΔU and $\Delta \Pi$:

$$\Delta W = \frac{\mu^2}{2}(\lambda_1 + 1)\lambda_2(2\lambda_1 + \lambda_1\lambda_2 + 2) + \frac{\tilde{\sigma}^2}{2}(3\lambda_2 + 1)$$

Dividing by μ^2, multiplying by 2, and rearranging, we obtain

$$\Delta W \propto \lambda_2(\lambda_2 + 2)\lambda_1^2 + \lambda_2(\lambda_2 + 4)\lambda_1 + (3\lambda_2 + 1)\tilde{\sigma}^2 + 2\lambda_2. \tag{52}$$

This is a quadratic expression in λ_2 with a coefficient $\lambda_1(1 + \lambda_1)$ on the quadratic term. The two roots are given by

$$\tilde{\lambda}_2(\lambda_1, \tilde{\sigma}) \triangleq \frac{-3\tilde{\sigma}^2 - 2(1 + \lambda_1)^2 + \sqrt{-4\tilde{\sigma}^2\lambda_1(1 + \lambda_1) + \left(3\tilde{\sigma}^2 + 2(1 + \lambda_1)^2\right)^2}}{2\lambda_1(1 + \lambda_1)},$$

and

$$\tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma}) \triangleq \frac{-3\tilde{\sigma}^2 - 2(1 + \lambda_1)^2 - \sqrt{-4\tilde{\sigma}^2\lambda_1(1 + \lambda_1) + \left(3\tilde{\sigma}^2 + 2(1 + \lambda_1)^2\right)^2}}{2\lambda_1(1 + \lambda_1)}.$$

The term in the root is always positive. Furthermore, the following properties hold.

Whenever $\lambda_1 \geq 0$, we have $0 > \tilde{\lambda}_2(\lambda_1, \tilde{\sigma}) > -1/2 > \tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma})$ for all $\tilde{\sigma} \geq 0$, and the expression (52) has a positive coefficient on the quadratic term. Therefore, all $\lambda_2 \geq \tilde{\lambda}_2(\lambda_1, \tilde{\sigma})$ increase social welfare.

Whenever $\lambda_1 < 0$, we have $-1/2 < \tilde{\lambda}_2(\lambda_1, \tilde{\sigma}) < 0 < \tilde{\tilde{\lambda}}_2(\lambda_1, \tilde{\sigma})$ and (52) has a negative coefficient on the quadratic term. Therefore, all $\lambda_2 \in [\tilde{\lambda}_2, \tilde{\tilde{\lambda}}_2]$ increase social welfare. □

In Figure 13, we illustrate the set of welfare-improving linkages (λ_1, λ_2).

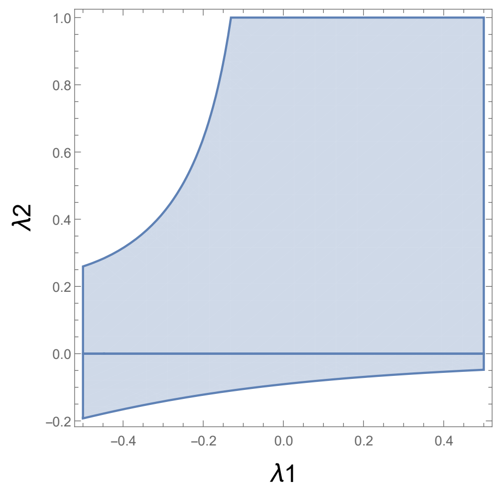

Figure 13: Socially Efficient Linkages $(\tilde{\sigma} = 1/2)$

In a static version of our model, the social value of information is positive for all λ larger than a threshold $\lambda^* < 0$. In a dynamic model with a data linkage, the consumer has an incentive to distort her demand, and the situation becomes more complex. Specifically, suppose the second-period firm has a large $\lambda_2 > 0$: if the first-period firm has $\lambda_1 < 0$, any linkage between these two firms causes a considerable loss in consumer surplus due to higher monopoly prices and upward quantity distortions in the first period. Likewise, for large $\lambda_1 > 0$, any linkage with $\lambda_2 < 0$ causes an inefficient reduction in consumer demand and underprovision of quality. The resulting loss is more severe for larger values of λ_1, for which the consumer's average consumption is higher. Thus, relative to the λ_2 cutoff policy of Proposition 3, the social planner forms all linkages such that (heuristically) $\lambda_1 \cdot \lambda_2$ is sufficiently large, and would only form linkages with $\lambda_2 > 0$ as λ_1 becomes large.

In this scenario, the consequences for consumer welfare relative to the outcome of regulation depend heavily on the distribution of bargaining power. In our stylized setting, where firm 1 has all the bargaining power, the consumer is as well off as under privacy for any (λ_1, λ_2). This outcome is weakly worse than that under required consent for any (λ_1, λ_2), but the ranking relative to *laissez faire*, transparency, or no discrimination is sensitive to the specific values of λ_1, λ_2, and $\tilde{\sigma}$.

G Consent by Informed Consumers

In this section, we revisit the most favorable privacy regulation in our baseline model, i.e., the case of Voluntary Consent (Section 5.3.2). We now analyze a game where consumers make their consent decisions after learning their first-period type. Our goal is to investigate the robustness of our conclusions regarding the active linkages in equilibrium under uninformed consent, which are given by $(\lambda_1, \lambda_2) \in \Lambda^{CS} \cap \Lambda^{PS}$ as in Proposition 9.

Our approach consists of characterizing pooling equilibria (when they exist) for any given pair of firms (λ_1, λ_2). Because in these equilibria all types θ grant or deny consent, these equilibria are outcome-equivalent to uninformed decisions by consumers in our baseline model.

For expositional convenience, we let $\varphi = 1$ and $\varepsilon_2 = 0$, i.e., $\theta_1 = \theta_2$, and we denote the consumer's type by just θ. We further impose a specific assumption on the distribution of the consumer's type, namely that θ is uniformly distributed over the interval

$$\theta \sim U\left[\bar{\theta}/2, \bar{\theta}\right], \tag{53}$$

parameterized by $\bar{\theta} > 0$. (Under this assumption, we show below that the existence of pooling equilibria is independent of $\overline{\theta}$ and is thus much easier to illustrate.)

We begin our analysis with the relevant subgame, in which the firms have offered a linkage to the consumer. In this subgame, we show that there always exists a pooling equilibrium without consent. In contrast, a pooling equilibrium with consent exists only for linkages in the set Λ^* defined by the inequalities in (56) at the bottom of this section.

Proposition 15 (Pooling Equilibria)**.**

1. *For any* (λ_1, λ_2)*, there exists an equilibrium where all types* θ *deny consent.*

2. *There exists an equilibrium where all types* θ *grant consent if and only if* $(\lambda_1, \lambda_2) \in \Lambda^*$.

The first part of Proposition 15 states that consumers can always be "trapped" into denying consent. On the equilibrium path, the two firms do not update their prior beliefs on the consumer's type. In contrast, the second part of Proposition 15 establishes that pooling equilibria where every consumer grants consent do not always exist. Indeed, if such an equilibrium exists, it can be supported by degenerate off-path beliefs. In particular, a consumer who denies consent will face the worst possible terms of trade in a static game in both periods. However, on the candidate equilibrium path, the consumer must play her dynamic best response to the firms' prices, which entails costly behavior distortions. When these equilibrium distortions are too high, the consumer is then willing to face worse terms of trade in both periods in order to avoid them. This is the case if, for example $\lambda_1 < 0$ and λ_2 is sufficiently large.

Proof of Proposition 15. (1.) We build a pooling equilibrium with no consent. To do so, it is sufficient to specify the off-path beliefs (if a consumer grants consent) in such a way that both firms assign probability one to the worst type for that transaction (i.e., $\bar{\theta}/2$ if $\lambda_t > 0$ and $\bar{\theta}$ if $\lambda_t < 0$). Because the firms' beliefs are degenerate, the consumer cannot signal her type through her purchase level either. Thus, the terms of trade she faces in each period are the optimal ones (for the firms holding those beliefs) in a static game. Therefore, she would be better off denying consent and facing the equilibrium terms of trade for an anonymous consumer in a static game.

(2.) Consider the consumer's decision to grant consent. If she grants consent, she receives the equilibrium surplus level

$$U^* (\theta, \mu, \lambda_1) + U (\theta, \theta, \lambda_2) . \tag{54}$$

If she denies consent, under our uniform distribution assumption (53), she receives the terms of trade for the worst type in each period. This type is given by

$$\hat{\theta} (\lambda) \triangleq \frac{\bar{\theta}}{2} + \mathbf{1}_{\{\lambda<0\}} \frac{\bar{\theta}}{2}.$$

Consequently, the consumer's surplus off-path is given by

$$U \left(\theta, \hat{\theta} (\lambda_1) , \lambda_1\right) + U \left(\theta, \hat{\theta} (\lambda_2) , \lambda_2\right) . \tag{55}$$

The resulting difference in surplus levels (54)-(55) can be written as a quadratic function of θ, with a coefficient of λ_2 on the term θ^2. Evaluating the difference at the endpoints of the support of the type distribution (if $\lambda_2 < 0$) or at the unique critical point (if $\lambda_2 > 0$), and substituting the definition of the worst type $\hat{\theta} (\lambda)$, we obtain the set of linkages $(\lambda_1, \lambda_2) \in \Lambda^*$ for which

$$U^* (\theta, \mu, \lambda_1) + U (\theta, \theta, \lambda_2) \geq U \left(\theta, \hat{\theta} (\lambda_1) , \lambda_1\right) + U \left(\theta, \hat{\theta} (\lambda_2) , \lambda_2\right)$$

for all θ. This set is given by the union of the regions in (λ_1, λ_2) space described in (56) and illustrated in Figure 14. Recall that the set Λ^* is independent of $\bar{\theta}$ under our distribution assumption (53). □

We now compare the set of linkages Λ^* with the set of linkages Λ^{CS} that benefit consumers ex ante. Although the two share similar qualitative properties, they are distinct. However, as $\tilde{\sigma} \to 0$, the set Λ^{CS} becomes a strict subset of Λ^*. This is shown in Figure 15.

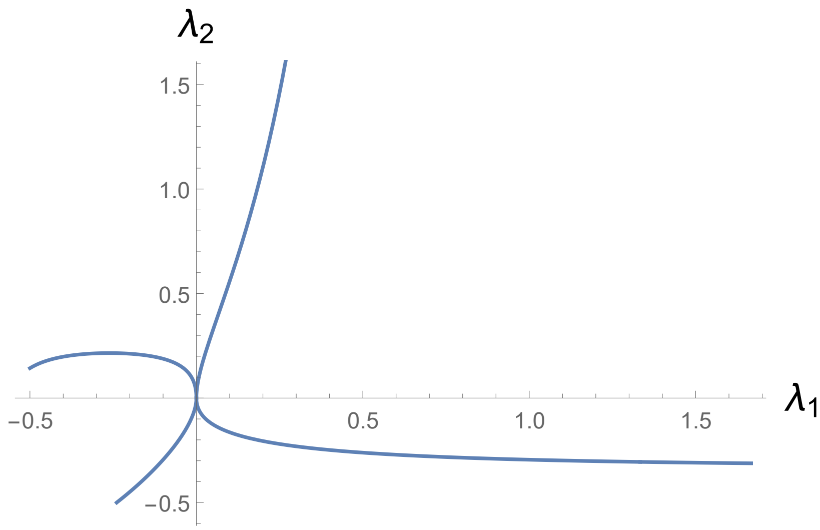

Figure 14: Linkages $(\lambda_1, \lambda_2) \in \Lambda^*$

A fortiori, as the degree of uncertainty over the consumer's type vanishes, the set of linkages that do form under ex ante consent $\Lambda^{CS} \cap \Lambda^{PS}$ is a strict subset of Λ^*. When this is the case, we can select the pooling equilibrium with consent for any $(\lambda_1, \lambda_2) \in \Lambda^{CS} \cap \Lambda^{PS}$ and the pooling equilibrium without consent for all other (λ_1, λ_2).

Corollary 3. *As $\tilde{\sigma} \to 0$, the equilibrium outcome of the game with uninformed voluntary consent can be obtained as a pooling equilibrium of the game with informed consent for all (λ_1, λ_2).*

Finally, note that there may exist other equilibria for some sets of linkages, such as threshold equilibria in which the consumer's types are partitioned into two intervals. In any one of these equilibria, high consumer types may grant or deny consent, depending on the firms' types. However, these equilibria exist for limited ranges of parameters, even in the uniform case.

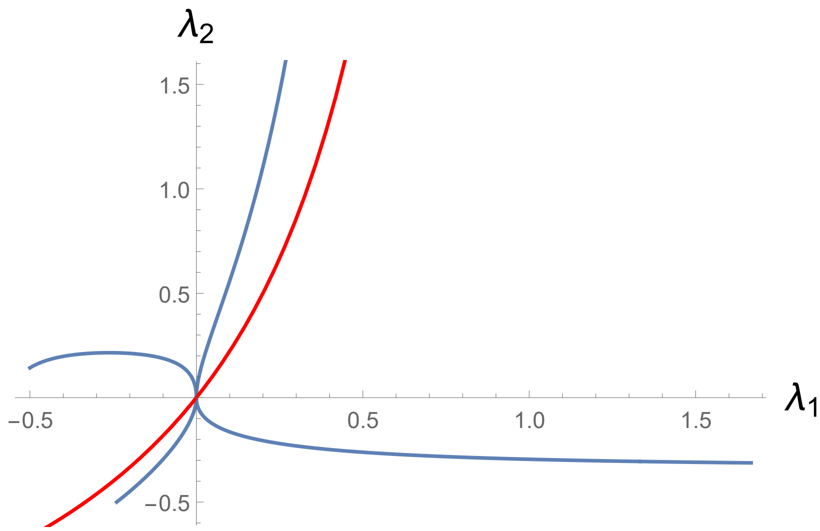

Figure 15: Comparison of Λ^* and Λ^{CS}

Note: the set Λ^* is given by the union of the following regions. (A Mathematica file with the calculations is available from the authors.)

$$\left\{\lambda_1 > \frac{4}{3}, \lambda_2 \in \left[\frac{2\sqrt{\frac{8\lambda_1+8\lambda_1^2+3\lambda_1^3}{4+3\lambda_1}} - 3\lambda_1}{3\lambda_1 - 4}, 0\right]\right\}$$
$$\left\{\lambda_1 \in [0, 4/3], \lambda_2 \in \left[-\frac{2\sqrt{\frac{8\lambda_1+8\lambda_1^2+3\lambda_1^3}{4+3\lambda_1}} - 3\lambda_1}{4 - 3\lambda_1}, 0\right]\right\}$$
$$\left\{\lambda_1 \in \left[-\frac{1}{2}, \frac{4}{55}\left(3\sqrt{5} - 10\right), \lambda_2 \in \left[-\frac{1}{2}, 0\right]\right]\right\}$$
$$\left\{\lambda_1 \in \left[\frac{4}{55}\left(3\sqrt{5} - 10\right), 0, \lambda_2 \in \left[-\frac{4\sqrt{\frac{2\lambda_1+2\lambda_1^2+3\lambda_1^3}{4+3\lambda_1}} - 3\lambda_1}{4 - 3\lambda_1}, 0\right]\right]\right\}$$
$$\left\{\lambda_1 > \frac{2}{3}, \lambda_2 > 0\right\}$$
$$\left\{\lambda_1 \in \left[0, \frac{2}{3}\right], \lambda_2 \in \left[0, \frac{3\lambda_1}{2 - 3\lambda_1} + 2\sqrt{\frac{3\lambda_1^3 + 4\lambda_1^2 + 2\lambda_1}{(3\lambda_1 - 2)^2 (2 + 3\lambda_1)}}\right]\right\}$$
$$\left\{\lambda_1 \in \left[-\frac{1}{2}, 0\right], \lambda_2 \in \left[0, \frac{3\lambda_1}{2 - 3\lambda_1} + 2\sqrt{2}\sqrt{\frac{6\lambda_1^3 + 2\lambda_1^2 - \lambda_1}{(3\lambda_1 - 2)^2 (2 + 3\lambda_1)}}\right]\right\}. \tag{56}$$