

Supplemental Appendix

Welfare of Competitive Price Discrimination with Captive Consumers

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A: Consumer-Optimal and Social-Optimal Segmentations

This subsection collects proofs for consumer-optimal and social-optimal segmentations. It is organized as follows. First, we restate Proposition 5 as Proposition A1 and provide a detailed proof. Second, in Proposition A2, we prove that a consumer-optimal segmentation yields a strictly higher producer surplus and hence leads to a Pareto improvement relative to uniform pricing, if and only if $l_1^C \in (\gamma_1/(1 - \gamma_2), 1)$. We characterize the social-optimal segmentation in Proposition A3, and investigate the issue of alignment between social surplus and consumer surplus in Proposition A4.

Proposition A1 *Suppose that $\gamma_1 > \gamma_2$ and consumers are highly risk-averse (i.e., condition (18) fails). Let $l_1^C \in (0, 1)$ be implicitly defined by $\Phi(l_1^C) = 0$.*

1. *If $l_1^C \in (0, \gamma_2/(1 - \gamma_1)]$, then the consumer-optimal segmentation is the standard field-leveling segmentation.*
2. *If $l_1^C \in (\gamma_2/(1 - \gamma_1), \gamma_1/(1 - \gamma_2)]$, then following modified field-leveling segmentation maximizes consumer surplus:*

$$\begin{aligned} & \frac{\gamma_2(1 + l_1^C)}{l_1^C} \left(\frac{l_1^C}{1 + l_1^C}, \frac{l_1^C}{1 + l_1^C} \right) + \frac{l_1^C(1 - \gamma_1) - \gamma_2}{l_1^C(1 - l_1^C)} (l_1^C, 0) \\ & + \frac{\gamma_1 - (1 - \gamma_2)l_1^C}{1 - l_1^C} (1, 0). \end{aligned} \tag{A1}$$

3. *If $l_1^C \in (\gamma_1/(1 - \gamma_2), 1)$, then following modified field-leveling segmentation maximizes consumer surplus:*

$$m_S^* \left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*} \right) + (1 - m_S^*) \left(\frac{\gamma_1 - \gamma_2}{1 - m_S^*}, 0 \right) \tag{A2}$$

where $m_S^* \in (2\gamma_2, \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1)$ is implicitly determined by the first-order condition of Problem (OPT-D). Moreover, this segmentation yields a strictly higher producer surplus than uniform pricing, leading to a Pareto improvement.

Proof. By Proposition 1, the C-Max problem is equivalent to Problem (OPT-D) with $A(q_1, q_2) = C(q_1, q_1)$:

$$\max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S C^\# \left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S} \right) + (1 - m_S) C^\# \left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0 \right).$$

By Lemma 4, $C^\#(q, q) = C(q, q)$. The objective of Problem (OPT-D) is strictly concave because its second-order derivative with respect to m_S is given by

$$\frac{\partial^2 C^\# \left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S} \right)}{\partial \left(\frac{\gamma_2}{m_S} \right)^2} \frac{\gamma_2^2}{m_S^3} + \frac{\partial^2 C^\# \left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0 \right)}{\partial \left(\frac{\gamma_1 - \gamma_2}{1 - m_S} \right)^2} \frac{(\gamma_1 - \gamma_2)^2}{(1 - m_S)^3} < 0,$$

where the inequality follows because $C^\#(\gamma_2/m_S, \gamma_2/m_S) = C(\gamma_2/m_S, \gamma_2/m_S)$ is strictly concave and $C^\#((\gamma_1 - \gamma_2)/(1 - m_S), 0)$ is weakly concave. Therefore, m_S^* is either a corner solution with $m_S^* \in \{2\gamma_2, 1 - \gamma_1 + \gamma_2\}$, or a unique interior solution to the first-order condition. Consequently, if there is an m_S^* such that it is interior and satisfies the first-order condition, then m_S^* solves Problem (OPT-D).

To further simplify the objective of Problem (OPT-D), we need Lemma 6, which shows that the concave envelope of a nested segment is given by

$$C^\#(q_1, 0) = \begin{cases} C(q_1, 0) & \text{if } q_1 \leq l_1^C \\ \frac{1-q_1}{1-l_1^C} C(l_1^C, 0) + \frac{q_1-l_1^C}{1-l_1^C} C(1, 0) & \text{if } q_1 > l_1^C \end{cases}$$

In order to apply Lemma 6, we consider the three cases separately (in the reverse order) in the remaining proof.

Case 3: $l_1^C \in (\gamma_1/(1 - \gamma_2), 1)$. We argue that there is an interior solution $m_S^* \in (2\gamma_2, \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1)$.

We first show that $m_S^* < \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$. Note that for $m_S \leq \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$, we have

$$\frac{\gamma_1 - \gamma_2}{1 - m_S} \leq \frac{\gamma_1 - \gamma_2}{1 - \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}} = \frac{\gamma_1}{1 - \gamma_2} < l_1^C$$

where the last inequality follows from assumption that $l_1^C \in (\gamma_1/(1 - \gamma_2), 1)$. Hence, by Lemma 6, $C^\#((\gamma_1 - \gamma_2)/(1 - m_S), 0) = C((\gamma_1 - \gamma_2)/(1 - m_S), 0)$, and the objective function of Problem (OPT-D) becomes

$$m_S C \left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S} \right) + (1 - m_S) C \left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0 \right). \quad (\text{A3})$$

The first-order derivative is

$$\begin{aligned} & -\frac{2\gamma_2^2}{(m_S - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_2}{m_S - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2(m_S - \gamma_2)}{m_S - 2\gamma_2} v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) \\ & + \frac{(\gamma_1 - \gamma_2)^2}{(1 - m_S)^2} v(\pi^*) + \frac{2(\gamma_1 - \gamma_2)^2}{(1 - m_S)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - m_S} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi - 2v\left(\frac{\gamma_1 - \gamma_2}{1 - m_S} \pi^*\right). \end{aligned}$$

It is negative at $m_S = \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$, because

$$\begin{aligned} & -2\frac{\gamma_1^2}{(\gamma_1 + \gamma_2 - 1)^2} (\pi^*)^2 \int_{\frac{\gamma_1}{1 - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2 - 2\gamma_2}{1 - \gamma_1 - \gamma_2} v\left(\frac{\gamma_1}{1 - \gamma_2} \pi^*\right) \\ & + \frac{\gamma_1^2}{(\gamma_2 - 1)^2} v(\pi^*) + \frac{2\gamma_1^2}{(\gamma_2 - 1)^2} (\pi^*)^2 \int_{\frac{\gamma_1}{1 - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi - 2v\left(\frac{\gamma_1}{1 - \gamma_2} \pi^*\right) \\ & = \frac{\gamma_1^2}{(\gamma_2 - 1)^2} v(\pi^*) - \frac{2\gamma_1^3(2 - \gamma_1 - 2\gamma_2)}{(\gamma_2 - 1)^2(\gamma_1 + \gamma_2 - 1)^2} (\pi^*)^2 \int_{\frac{\gamma_1}{1 - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2\gamma_1 v\left(\frac{\gamma_1}{1 - \gamma_2} \pi^*\right)}{1 - \gamma_1 - \gamma_2} \\ & = \frac{\gamma_1^2}{(\gamma_1 + \gamma_2 - 1)^2} \Phi\left(\frac{\gamma_1}{1 - \gamma_2}\right) < 0 \end{aligned}$$

where the last equality follows from the definition of l_1^C and the assumption of $\gamma_1/(1 - \gamma_2) < l_1^C$. Since the objective of Problem (OPT-D) is continuous and strictly concave in m_S , its derivative must be strictly negative for all $m_S > \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$. It follows that the optimal solution m_S^* must satisfy $m_S^* < \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$.

Next we argue that $m_S^* > 2\gamma_2$. It suffices to show that the first-order derivative is strictly positive at $m_S = 2\gamma_2$. Since $m_S^* < \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$, we have $(\gamma_1 - \gamma_2)/(1 - m_S) < l_1^C$. The objective of Problem (OPT-D) is then given in (A3). Hence, the first-order derivative evaluated at $m_S = 2\gamma_2$ is

$$\begin{aligned} & \lim_{m_S \rightarrow 2\gamma_2} \left(-\frac{2\gamma_2^2 (\pi^*)^2}{(m_S - 2\gamma_2)^2} \int_{\frac{\gamma_2}{m_S - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2m_S - 2\gamma_2}{m_S - 2\gamma_2} v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) \right) \\ & + \frac{(\gamma_1 - \gamma_2)^2}{(1 - m_S)^2} v(\pi^*) + \frac{2(\gamma_1 - \gamma_2)^2}{(1 - m_S)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - m_S} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi - 2v\left(\frac{\gamma_1 - \gamma_2}{1 - m_S} \pi^*\right). \end{aligned}$$

To show it is strictly positive, we first note that by L'Hospital's rule,

$$\begin{aligned}
& \lim_{m_S \rightarrow 2\gamma_2} \left(-\frac{2\gamma_2^2 (\pi^*)^2}{(m_S - 2\gamma_2)^2} \int_{\frac{\gamma_2}{m_S - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2m_S - 2\gamma_2}{m_S - 2\gamma_2} v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) \right) \\
&= \lim_{m_S \rightarrow 2\gamma_2} \left(\frac{(2m_S - 2\gamma_2)(m_S - 2\gamma_2)v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) - 2\gamma_2^2 (\pi^*)^2 \int_{\frac{\gamma_2}{m_S - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi}{(m_S - 2\gamma_2)^2} \right) \\
&= \lim_{m_S \rightarrow 2\gamma_2} \left(\frac{2(m_S - 2\gamma_2)v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) - \frac{2\gamma_2(m_S - 2\gamma_2)}{m_S - \gamma_2} v'\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) \pi^*}{2(m_S - 2\gamma_2)} \right) \\
&= v(\pi^*) - v'(\pi^*) \pi^*. \tag{A4}
\end{aligned}$$

We further note that

$$\begin{aligned}
& \frac{(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} v(\pi^*) + \frac{2(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi - 2v\left(\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*\right) \\
&= \frac{(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} \frac{v'(\pi)}{\pi^2} d\pi - v\left(\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*\right). \tag{A5}
\end{aligned}$$

We can use (A4) and (A5) to rewrite the first-order derivative at $m_S = 2\gamma_2$ as

$$\begin{aligned}
& \frac{(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} \frac{v'(\pi)}{\pi^2} d\pi - v\left(\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*\right) + v(\pi^*) - v'(\pi^*) \pi^* \\
&= \frac{(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} \frac{v'(\pi)}{\pi^2} d\pi + \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} v'(\pi) d\pi - v'(\pi^*) \pi^* \\
&\geq \frac{(\gamma_1 - \gamma_2)^2}{(1 - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} \frac{v'(\pi^*)}{\pi^2} d\pi + \int_{\frac{\gamma_1 - \gamma_2}{1 - 2\gamma_2} \pi^*}^{\pi^*} v'(\pi^*) d\pi - v'(\pi^*) \pi^* \\
&= -\frac{(\gamma_1 - \gamma_2)^2}{(2\gamma_2 - 1)^2} v'(\pi^*) \pi^* \\
&> 0,
\end{aligned}$$

where the first inequality follows since $\pi'(\pi) \geq \pi'(\pi^*)$ for all $\pi < \pi^*$ by concavity of $v(\pi)$. Given that $l_1^C < 1$, $(\gamma_1 - \gamma_2)/(1 - m_S^*) < l_1^C$ implies that $m_S^* < 1 - \gamma_1 + \gamma_2$. Therefore, we must have $m_S^* \in (2\gamma_2, 1 - \gamma_1 + \gamma_2)$ and it is implicitly determined by the first-order condition.

The statement about Pareto improvement is a direct corollary of Proposition A2 below. This completes the proof for Case 3.

Case 2: $l_1^C \in (\gamma_2/(1 - \gamma_1), \gamma_1/(1 - \gamma_2)]$. We guess and verify that there is an interior optimal solution $m_S^* = \gamma_2(1 + l_1^C)/l_1^C$. We first verify that m_S^* satisfies the first-order

condition. Note that, at $m_S = \gamma_2 (1 + l_1^C) / l_1^C$,

$$\frac{\gamma_1 - \gamma_2}{1 - m_S} = \frac{\gamma_1 - \gamma_2}{1 - \frac{\gamma_2(1+l_1^C)}{l_1^C}} = \frac{\gamma_1 - \gamma_2}{(1 - \gamma_2)l_1^C - \gamma_2} l_1^C \geq l_1^C$$

where the inequality follows from the case assumption of $\gamma_1/(1 - \gamma_2) \geq l_1^C$. Therefore, the objective function can be rewritten as

$$m_S C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S) \left(\frac{1 - \frac{\gamma_1 - \gamma_2}{1 - m_S}}{1 - l_1^C} C(l_1^C, 0) + \frac{\frac{\gamma_1 - \gamma_2}{1 - m_S} - l_1^C}{1 - l_1^C} C(1, 0) \right).$$

Substituting $m_S = \gamma_2 (1 + l_1^C) / l_1^C$ into the first-order derivative, we have

$$\begin{aligned} & -\frac{2\gamma_2^2}{(m_S - 2\gamma_2)^2} (\pi^*)^2 \int_{\frac{\gamma_2}{m_S - \gamma_2} \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2(m_S - \gamma_2)}{m_S - 2\gamma_2} v\left(\frac{\gamma_2}{m_S - \gamma_2} \pi^*\right) \\ & + \frac{l_1^C C(1, 0) - C(l_1^C, 0)}{1 - l_1^C} \\ = & -2 \frac{(l_1^C)^2}{(l_1^C - 1)^2} (\pi^*)^2 \int_{l_1^C \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2}{1 - l_1^C} v(l_1^C \pi^*) + \frac{l_1^C C(1, 0) - C(l_1^C, 0)}{1 - l_1^C} \\ = & -2 \frac{(l_1^C)^2}{(l_1^C - 1)^2} (\pi^*)^2 \int_{l_1^C \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2}{1 - l_1^C} v(l_1^C \pi^*) \\ & + (l_1^C)^2 v(\pi^*) + 2(l_1^C)^2 (\pi^*)^2 \int_{l_1^C \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi - 2v(l_1^C \pi^*) \\ = & (l_1^C)^2 v(\pi^*) - 2 \frac{(l_1^C)^3 (2 - l_1^C)}{(l_1^C - 1)^2} (\pi^*)^2 \int_{l_1^C \pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2l_1^C}{1 - l_1^C} v(l_1^C \pi^*) \\ = & \frac{(l_1^C)^2}{(1 - l_1^C)^2} \Phi(l_1^C) = 0. \end{aligned}$$

It remains to show that m_S^* is interior. Since, by the case assumption, $l_1^C \in (\gamma_2/(1 - \gamma_1), 1)$, we have $\gamma_2(1 + l_1^C)/l_1^C \in (2\gamma_2, 1 - \gamma_1 + \gamma_2)$. Therefore, the optimal m_S^* is indeed $m_S^* = \gamma_2(1 + l_1^C)/l_1^C$, and an optimal segmentation is as specified in the proposition.

Case 1: $l_1^C \in (0, \gamma_2/(1 - \gamma_1)]$. Similarly to Case 2, the first-order derivative equals 0 at $m_S = \gamma_2(1 + l_1^C)/l_1^C$. By the case assumption, $\gamma_2(1 + l_1^C)/l_1^C \geq 1 - \gamma_1 + \gamma_2$. Since the objective of Problem (OPT-D) is strictly concave, the objective is strictly increasing in m_S for all $m_S \in [2\gamma_2, 1 - \gamma_1 + \gamma_2)$. Thus, the optimal solution to Problem (OPT-D) is $m_S^* = 1 - \gamma_1 + \gamma_2$ and the standard field-leveling segmentation is optimal. ■

The optimal solution is unique in Cases 1 and 3, but is not unique in Case 2. In Case 2, we can replace any segment whose captive-to-reach ratio equals l_1^C with infinite

combinations of segments that have the same captive-to-reach ratio. By Lemma 2, all these combinations yield the same producer surplus. Indeed, in Case 2, all consumer-optimal segmentations generate the same producer and consumer surplus.

The following proposition shows that the consumer-optimal segmentation in Case 3 strictly increases producer surplus relative to uniform pricing, and hence it leads to a Pareto improvement. In fact, it provides a stronger result—it characterizes the necessary and sufficient condition for Pareto improvement.

Proposition A2 *Suppose $\gamma_1 > \gamma_2$. A consumer-optimal segmentation yields a strictly higher producer surplus than uniform pricing if and only if condition (18) fails and $\gamma_1/(1 - \gamma_2) < l_1^C$.*

Proof. Necessity. Since field-leveling segmentation minimizes producer surplus, the failure of (18) is clearly necessary, and consumer-optimal segmentation cannot increase producer surplus in Case 1 of Proposition A1. Therefore, to show that $\gamma_1/(1 - \gamma_2) < l_1^C$ is also necessary, it suffices to show that the optimal segmentation (A1) in Case 2 does not strictly increase producer surplus relative to uniform pricing. By Lemma 2, the segmentation

$$\frac{\gamma_2(1 + \gamma_1 - \gamma_2)}{\gamma_1} \left(\frac{\gamma_1}{1 + \gamma_1 - \gamma_2}, \frac{\gamma_1}{1 + \gamma_1 - \gamma_2} \right) + \frac{(\gamma_1 - \gamma_2)(1 - \gamma_2)}{\gamma_1} \left(\frac{\gamma_1}{1 - \gamma_2}, 0 \right) \quad (\text{A6})$$

yields the same producer surplus as uniform pricing. This segmentation will serve as our benchmark in evaluating whether the optimal segmentation (A1) in Case 2 can strictly improve producer surplus.

Since $P(q, q)$ is linear in q , the consumer-optimal segmentation (A1) yields the same producer surplus as the segmentation

$$\begin{aligned} & \frac{\gamma_2(1 + \gamma_1 - \gamma_2)}{\gamma_1} \left(\frac{\gamma_1}{1 + \gamma_1 - \gamma_2}, \frac{\gamma_1}{1 + \gamma_1 - \gamma_2} \right) \\ & + \left(\frac{\gamma_2(1 + l_1^C)}{l_1^C} - \frac{\gamma_2(1 + \gamma_1 - \gamma_2)}{\gamma_1} \right) (0, 0) \\ & + \frac{l_1^C - l_1^C \gamma_1 - \gamma_2}{l_1^C(1 - l_1^C)} (l_1^C, 0) + \frac{\gamma_1 + l_1^C \gamma_2 - l_1^C}{1 - l_1^C} (1, 0). \end{aligned}$$

Because $P(q_1, 0)$ is strictly concave in q_1 , we can weakly increase the producer surplus by replacing the last three components of this segmentation with $(\gamma_1 - \gamma_2)(1 - \gamma_2)/\gamma_1 * (\gamma_1/(1 - \gamma_2), 0)$, which leads to the segmentation in (A6). Therefore, producer surplus is lower under segmentation (A1) than under uniform pricing. This completes the proof of necessity.

Sufficiency. It suffices to show that the consumer-optimal segmentation (A2) in Proposition A1 strictly increases producer surplus relative to segmentation (A6). Since $m_S^* < \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$, it is sufficient to show that the following problem is strictly decreasing in m_S :

$$\max_{m_S \in \left[2\gamma_2, \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}\right]} m_S P\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S) P\left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0\right).$$

Because $P(\gamma_1, \gamma_2) = (2 - \gamma_1 - \gamma_2)\gamma_1\pi^*/(1 - \gamma_2)$, the objective function is

$$2\gamma_2\pi^* + \left(2 - \frac{\gamma_1 - \gamma_2}{1 - m_S}\right)(\gamma_1 - \gamma_2)\pi^*$$

which is clearly strictly decreasing in m_S . This completes the proof for sufficiency. ■

Finally, we investigate when consumer surplus and social surplus are aligned in the sense that consumer-optimal segmentation also maximizes social surplus. We first solves the S-Max problem.

As we have argued in the text, $s(\pi^*) - s(0) - 2\pi^*s'(0) < 0$. Therefore, by Lemma 6,

$$S^\#(q_1, 0) = \begin{cases} S(q_1, 0) & \text{if } q_1 \leq l_1^S \\ \frac{1-q_1}{1-l_1^S} S(l_1^S, 0) + \frac{q_1-l_1^S}{1-l_1^S} S(1, 0) & \text{if } q_1 > l_1^S \end{cases}$$

where $l_1^S \in (0, 1)$ is implicitly defined by $\Phi^S(l_1^S) = 0$ with

$$\Phi^S(l) = (1 - l)^2 s(\pi^*) - 2l(2 - l)(\pi^*)^2 \int_{l\pi^*}^{\pi^*} \frac{s(\pi)}{\pi^3} d\pi + \frac{2(1 - l)s(l\pi^*)}{l}.$$

Note that $l_1^S \neq l_1^C$.

By Proposition 1, the S-Max problem is equivalent to Problem (OPT-D) with $A(q_1, q_2) = S(q_1, q_1)$:

$$\max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S S^\# \left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S} \right) + (1 - m_S) S^\# \left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0 \right).$$

By Lemma 4, $S^\#(q, q) = S(q, q)$.

The following proposition characterize the S-Max segmentation. Its proof is similar to that of Proposition A1 and hence omitted.

Proposition A3 *Suppose that $\gamma_1 > \gamma_2$ and let $l_1^S \in (0, 1)$ be implicitly defined by $\Phi(l_1^S) = 0$.*

1. If $l_1^S \in (0, \gamma_2/(1-\gamma_1)]$, then the social-optimal segmentation is the standard field-leveling segmentation.
2. If $l_1^S \in (\gamma_2/(1-\gamma_1), \gamma_1/(1-\gamma_2)]$, then following modified field-leveling segmentation maximizes social surplus:

$$\begin{aligned} & \frac{\gamma_2(1+l_1^S)}{l_1^S} \left(\frac{l_1^S}{1+l_1^S}, \frac{l_1^S}{1+l_1^S} \right) + \frac{l_1^S(1-\gamma_1)-\gamma_2}{l_1^S(1-l_1^S)} (l_1^S, 0) \\ & + \frac{\gamma_1-(1-\gamma_2)l_1^S}{1-l_1^S} (1, 0). \end{aligned} \quad (\text{A7})$$

3. If $l_1^S \in (\gamma_1/(1-\gamma_2), 1)$, then following modified field-leveling segmentation maximizes social surplus:

$$m_S^S \left(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S} \right) + (1-m_S^S) \left(\frac{\gamma_1-\gamma_2}{1-m_S^S}, 0 \right) \quad (\text{A8})$$

where $m_S^S \in (2\gamma_2, \gamma_2(1+\gamma_1-\gamma_2)/\gamma_1)$ is implicitly determined by the first-order condition of Problem (OPT-D).

Now we are ready to investigate when the social-optimal segmentation coincides with the consumer-optimal segmentation.

Proposition A4 Suppose $\gamma_1 > \gamma_2$. Consumer surplus and social surplus are aligned if and only if $\gamma_2/(1-\gamma_1) \geq l_1^S$.

Proof. We distinguish two cases.

Case (i): Condition (18) holds. By Proposition 4, field-leveling segmentation is consumer-optimal. Hence, consumer- and social-optimal segmentations coincide if and only if the conditions in Case 1 of Proposition A3 are satisfied, or equivalently if $\gamma_2/(1-\gamma_1) \geq l_1^S$.

Case (ii): Condition (18) fails. In this case, the consumer-optimal segmentation is given by Proposition A1. Note first that m_S^* in Proposition A1 is determined by

$$C \left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*} \right) - C \left(\frac{\gamma_1-\gamma_2}{1-m_S^*}, 0 \right) - \frac{\gamma_2}{m_S^*} \frac{\partial C(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*})}{\partial \frac{\gamma_2}{m_S^*}} + \frac{\gamma_1-\gamma_2}{1-m_S^*} \frac{\partial C(\frac{\gamma_1-\gamma_2}{1-m_S^*}, 0)}{\partial \frac{\gamma_1-\gamma_2}{1-m_S^*}} = 0$$

and m_S^S in Proposition A3 is determined by

$$\begin{aligned}
0 &= S\left(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S}\right) - S\left(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0\right) - \frac{\gamma_2}{m_S^S} \frac{\partial S(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S})}{\partial \frac{\gamma_2}{m_S^S}} + \frac{\gamma_1 - \gamma_2}{1 - m_S^S} \frac{\partial S(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0)}{\partial \frac{\gamma_1 - \gamma_2}{1 - m_S^S}} \\
&= \frac{1}{2} \left(C\left(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S}\right) - C\left(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0\right) + P\left(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S}\right) - P\left(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0\right) \right) \\
&\quad - \frac{\gamma_2}{2m_S^S} \left(\frac{\partial C(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S})}{\partial \frac{\gamma_2}{m_S^S}} + \frac{\partial P(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S})}{\partial \frac{\gamma_2}{m_S^S}} \right) + \frac{\gamma_1 - \gamma_2}{2(1 - m_S^S)} \left(\frac{\partial C(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0)}{\partial \frac{\gamma_1 - \gamma_2}{1 - m_S^S}} + \frac{\partial P(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0)}{\partial \frac{\gamma_1 - \gamma_2}{1 - m_S^S}} \right) \\
&< \frac{1}{2} \left(C\left(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S}\right) - C\left(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0\right) \right) - \frac{\gamma_2}{2m_S^S} \frac{\partial C(\frac{\gamma_2}{m_S^S}, \frac{\gamma_2}{m_S^S})}{\partial \frac{\gamma_2}{m_S^S}} + \frac{\gamma_1 - \gamma_2}{2(1 - m_S^S)} \frac{\partial C(\frac{\gamma_1 - \gamma_2}{1 - m_S^S}, 0)}{\partial \frac{\gamma_1 - \gamma_2}{1 - m_S^S}}
\end{aligned}$$

where the the inequality holds because $P(0, 0) = 0$, $P(q_1, q_1)$ is linear and $P(q_1, 0)$ is strictly concave. Thus, m_S^S is different from m_S^* . Moreover,

$$\Phi^S(l_1^S) = \Phi(l_1^S) - (1 - l_1^S)^2 \pi^*.$$

which implies $l_1^S > l_1^C$. Thus, consumer- and social-optimal segmentations coincide if and only if the conditions in Case 1 of Proposition A1 and those in Case 1 of Proposition A3 both hold, i.e.

$$\frac{\gamma_2}{1 - \gamma_1} \geq l_1^C, \text{ and } \frac{\gamma_2}{1 - \gamma_1} \geq l_1^S.$$

Since $l_1^S > l_1^C$, this reduces to $\gamma_2/(1 - \gamma_1) \geq l_1^S$.

Combining the two cases, we conclude that if and only if

$$\frac{\gamma_2}{1 - \gamma_1} \geq l_1^S,$$

consumer surplus and social surplus are aligned. ■

B: Proofs of Lemma 5 and 6

This appendix contains ommitted proofs of Lemma 5 and 6.

Proof of Lemma 5. We first prove the monotonicity of $A(q_1, 0)$. If $a(\pi)$ is strictly

increasing, then $A(q_1, 0)$ is also strictly increasing because

$$\begin{aligned}
\frac{\partial A(q_1, 0)}{\partial q_1} &= 2q_1 a(\pi^*) + 4q_1 (\pi^*)^2 \int_{q_1 \pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2 \frac{a(q_1 \pi^*)}{q_1} \\
&> 2q_1 a(\pi^*) + 4q_1 (\pi^*)^2 \int_{q_1 \pi^*}^{\pi^*} \frac{a(q_1 \pi^*)}{\pi^3} d\pi - 2 \frac{a(q_1 \pi^*)}{q_1} \\
&= 2q_1 a(\pi^*) + 4q_1 (\pi^*)^2 \frac{1 - q_1^2}{2q_1^2 (\pi^*)^2} a(q_1 \pi^*) - 2 \frac{a(q_1 \pi^*)}{q_1} \\
&= 2q_1 (a(\pi^*) - a(q_1 \pi^*)) \\
&> 0.
\end{aligned}$$

The proof for the case with strictly decreasing $a(\pi)$ is symmetric.

Next, we prove the shape of $A(q_1, 0)$. Consider first Case (d-sav) where $a(\pi)$ is strictly decreasing and strictly concave. The second and third derivatives of $A(q_1, 0)$ are

$$\begin{aligned}
\frac{\partial^2 A(q_1, 0)}{\partial q_1^2} &= 2a(\pi^*) + 4(\pi^*)^2 \int_{q_1 \pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - \frac{2}{q_1^2} a(q_1 \pi^*) - \frac{2\pi^*}{q_1} a'(q_1 \pi^*) \\
\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} &= -\frac{2(\pi^*)^2}{q_1} a''(q_1 \pi^*)
\end{aligned}$$

It is immediate that

$$\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} > 0 \quad \text{and} \quad \lim_{q_1 \rightarrow 1} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) > 0.$$

To establish the claim that $A(q_1, 0)$ is first strictly concave and then strictly convex, it is sufficient to show that $\lim_{q_1 \rightarrow 0^+} \partial^2 A(q_1, 0)/\partial q_1^2 = -\infty$. Suppose by contradiction that $\lim_{q_1 \rightarrow 0^+} \partial^2 A(q_1, 0)/\partial q_1^2 = y \in (-\infty, \infty)$. Then we can repeatedly apply the L'Hôpital's rule to rewrite y as

$$\begin{aligned}
&2a(\pi^*) + \lim_{q_1 \rightarrow 0^+} \frac{4(\pi^*)^2 q_1^2 \int_{q_1 \pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2a(q_1 \pi^*) - 2\pi^* q_1 a'(q_1 \pi^*)}{q_1^2} \\
&= 2a(\pi^*) \\
&+ \lim_{q_1 \rightarrow 0^+} \frac{8(\pi^*)^2 q_1 \int_{q_1 \pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 4(\pi^*)^3 \frac{a(q_1 \pi^*)}{(q_1 \pi^*)^3} - 4\pi^* a'(q_1 \pi^*) - 2(\pi^*)^2 q_1 a''(q_1 \pi^*)}{2q_1} \\
&= 2a(\pi^*) + \lim_{q_1 \rightarrow 0^+} \frac{4(\pi^*)^2 q_1^2 \int_{q_1 \pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2a(q_1 \pi^*) - 2\pi^* q_1 a'(q_1 \pi^*)}{q_1^2} - (\pi^*)^2 a''(0) \\
&= y - (\pi^*)^2 a''(0),
\end{aligned}$$

which leads to a contradiction. Therefore, $\lim_{q_1 \rightarrow 0^+} \partial^2 A(q_1, 0)/\partial q_1^2$ does not have a

finite limit. But since $\partial^2 A(q_1, 0)/\partial q_1^2$ is continuous and strictly increasing in q_1 for all $q_1 \in (0, 1)$, we must have $\lim_{q_1 \rightarrow 0^+} \partial^2 A(q_1, 0)/\partial q_1^2 = -\infty$.

In Case (d-wcvx), $a(\pi)$ is strictly decreasing and weakly convex, and

$$\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} \leq 0 \quad \text{and} \quad \lim_{q_1 \rightarrow 1} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) > 0.$$

It follows that

$$\frac{\partial^2 A(q_1, 0)}{\partial q_1^2} > 0 \quad \text{for all } q_1 \in [0, 1],$$

and therefore $A(q_1, 0)$ is strictly convex in q_1 for all $q_1 \in [0, 1]$.

In Case (i-wcav), $a(\pi)$ is strictly increasing and weakly concave, and

$$\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} \geq 0 \quad \text{and} \quad \lim_{q_1 \rightarrow 1} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

This implies that

$$\frac{\partial^2 A(q_1, 0)}{\partial q_1^2} < 0 \quad \text{for all } q_1 \in [0, 1],$$

which means that $A(q_1, 0)$ is strictly concave.

Finally, for Case (i-scvx), since $a(\pi)$ is strictly increasing and strictly convex, it is immediate that

$$\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} < 0 \quad \text{and} \quad \lim_{q_1 \rightarrow 1} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

As we have shown in the proof of Case (d-scav), $\partial^2 A(q_1, 0)/\partial q_1^2$ does not have a finite limit as $q_1 \rightarrow 0^+$. We must have

$$\lim_{q_1 \rightarrow 0^+} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = +\infty.$$

Therefore, $A(q_1, 0)$ must be initially convex and then concave. ■

Proof of Lemma 6. For Case (d-scav), it remains to show that $l_1 = 0$, or equivalently

$\Phi(0) \geq 0$, if and only if $a(\pi^*) - a(0) - 2\pi^*a'(0) \geq 0$. This is true because

$$\begin{aligned}
\Phi(0) &= a(\pi^*) - 2 \lim_{l \rightarrow 0^+} \frac{(\pi^*)^2 \int_{l\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - \left(\frac{1-l}{l^2(2-l)}\right) a(l\pi^*)}{\frac{1}{l(2-l)}} \\
&= a(\pi^*) - 2 \lim_{l \rightarrow 0^+} \frac{-\frac{a(l\pi^*)}{l^3} - \frac{1-l}{l^2(2-l)} \pi^* a'(l\pi^*) + \frac{(2l^2-5l+4)}{l^3(2-l)^2} a(l\pi^*)}{-\frac{2-2l}{l^2(2-l)^2}} \\
&= a(\pi^*) - 2 \lim_{l \rightarrow 0^+} \frac{-\frac{1-l}{l^2(2-l)^2} a(l\pi^*) - \frac{1-l}{l^2(2-l)} \pi^* a'(l\pi^*)}{-\frac{2-2l}{l^2(2-l)^2}} \\
&= a(\pi^*) - \lim_{l \rightarrow 0^+} (a(l\pi^*) + (2-l) \pi^* a'(l\pi^*)) \\
&= a(\pi^*) - a(0) - 2\pi^* a'(0).
\end{aligned}$$

When l_1 is interior, the expression of concave envelope $A^\#(q_1, 0)$ directly follows from the definition of l_1 (see Figure 3).

The claims for Case (d-wcvx) and Case (i-wcav) directly follow from Lemma 5.

For Case (i-scvx), we first establish that l_2 is interior. In the proof of Lemma 5, we have shown that in Case (i-scvx)

$$\frac{\partial^3 A(q_1, 0)}{\partial q_1^3} < 0 \quad \text{and} \quad \lim_{q_1 \rightarrow 1} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

and that

$$\lim_{q_1 \rightarrow 0^+} \frac{\partial^2 A(q_1, 0)}{\partial q_1^2} = +\infty.$$

Therefore, there must exist some $l^* \in (0, 1)$ such that $\partial^2 A(q_1, 0)/\partial q_1^2 > 0$ if and only if $q_1 < l^*$. We can use the expressions for $A(q_1, 0)$ and $\partial A(q_1, 0)/\partial q_1$ to show that

$$\begin{aligned}
&A(l^*, 0) - 2a(l\pi^*) + a(0) \\
&= l^* \frac{\partial A(l_1^*, 0)}{\partial q_1} - A(l^*, 0) + A(0, 0) \\
&= l^* \frac{\partial A(l_1^*, 0)}{\partial q_1} - \int_0^{l^*} \frac{\partial A(q_1, 0)}{\partial q_1} dq_1 \\
&= \int_0^{l^*} \left(\frac{\partial A(l^*, 0)}{\partial q_1} - \frac{\partial A(q_1, 0)}{\partial q_1} \right) dq_1 \\
&> 0,
\end{aligned}$$

where the inequality follows from the convexity of $A(q_1, 0)$ for $q_1 < l^*$.

Since $A(l, 0) - 2a(l\pi^*) + a(0)$ is continuous in l and

$$A(1, 0) - 2a(\pi^*) + a(0) = -a(\pi^*) + a(0) < 0,$$

l_2 exists and $l_2 \in (0, 1)$. Then the expression of concave envelope $A^\#(q_1, 0)$ directly follows from the definition of l_2 (see Figure 3). ■