Supplemental Appendix

The Combinatorial Multi-Round Ascending Auction

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SA.1 Details of the 2019 Danish auction

Table SA.1 summarizes the supply in the 2019 auction and the auction outcome. The licenses in the 900 MHz band were not allocated through a CMRA. In the 900 MHz band, there were 2x30 MHz paired frequencies available. These licenses came with a coverage obligation. We call the lots in the 900 MHz band A lots.

The supply in the CMRA was six 2x5 MHz blocks (B lots) of paired frequencies in the 700 MHz band, four 5 MHz blocks of unpaired frequencies in the 700 MHz band (D lots), one block 40 MHz block in the 2.3 GHz band with a coverage obligation (E lot), and six 10 MHz blocks in the 2.3 GHz band (F lots).

There was no reserve price on the lots with a coverage obligation. The reserve price per B lot was DKK 95 million. The reserve price per D lot was DKK 25 million. The reserve price per F lot was DKK 25 million.

The following spectrum caps were in place. Each bidder was allowed to win at most one block in the 900 MHz band. Across the paired blocks in the 700 MHz and 900 MHz bands, each bidder was allowed to win at most four lots. Bidders were not allowed to win more than 60 MHz in the 2.3 GHz band. There was no restriction on the number of blocks a bidder could win in the unpaired 700 MHz band.

The auction outcome was as follows. Bidder A paid DKK 485.2 million for one A lot and two B lots. Bidder B paid DKK 1620 million for one A lot, three B lots, four D lots, and six F lots. Bidder C paid DKK 107.6 million for one A lot and one B lot. Hence, the 40 MHz lot in the 2.3 GHz spectrum with the coverage obligation was unsold. Note that bidder B received the maximum quantity permitted by the spectrum caps, which is consistent with the CMRA-truthful and constant strategies. For this reason, it is likely that bidder B won with their headline demand.

We now examine whether the bidders paid linear prices. Recall this would suggest that only headline demands were winning. As the A lots were traded at a reserve price of 0, bidder C paid DKK 107.6 million for a single B lot. Bidder A paid DKK 485.2 million for two B lots. As bidder A paid more than four times bidder C's payment, we take this as evidence that at least one additional bid was winning. In particular, we speculate that bidder C won with an additional bid. Winning a small package at low cost is consistent with the constant strategy and with CMRA-truthful bidding.

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
B	2x5 MHz in the 700 MHz band	6	95	2	3	1
D	5 MHz in the 700 MHz band	4	25		4	
E	40 MHz in the 2.3 GHz band	1	0			
F	10 MHz in the 2.3 GHz band	6	25		6	
Non-competitive						
A	2x10 MHz in the 900 MHz band	3	0	1	1	1
Expenditure						
	In million DKK		820	485	1620	107.6

Table SA.1: Supply and auction outcome in the 2019 Danish spectrum auction
Notes: R = reserve price in million DKK; the E lot came with a coverage obligation

While we think it is likely that bidder B won with a headline demand, it is not clear whether bidder A won with a headline or an additional bid. Bidder B paid DKK 1135 million more than bidder A for also winning another B lot, the four D lots, and the six F lots. If bidder B won the headline demand, then $1135 = p_B + 4p_D + 6p_F$. Suppose bidder A won the headline demand, implying $p_B = 485.2/2 = 242.6$ and $892.4 = 4p_D + 6p_F$. Moreover, assume that $p_D = p_F$ as the D and F lots have the same reserve price. Linear prices then imply that $p_D = p_F = 89.24$, which is not implausible. Conversely, if bidder B won their headline demand and final prices for D and F were, say, about 65, then this would imply $p_B = 485$. In particular, bidder A would have bought two B lots with an additional bid at half price. Hence, we cannot rule out the possibilities that bidder A won their headline demand or with an additional bid. Finally, we do not see any signs that the auctioned ended as in the risk-free demand reduction equilibrium.

SA.2 Details of the 2021 Danish auction

The process was similar to the two previous auctions. Bidders first had the chance to obtain 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0 (2.1-D lot). All bidders bought such a license. Table SA.2 summarizes the supply and outcome.

There were two subsequent CMRAs. In the first CMRA, there were ten lots in the 1500 MHz band available: a single 25 MHz (lot 1.5-B) for a reserve price of DKK 10 million, eight 5 MHz lot (1.5-M) for a reserve price of DKK 10 million each, and another single 25 MHz block (1.5-T) for a reserve price of DKK 10 million. In the 2.1 GHz spectrum, there were six 2x5 MHz blocks (2.1-U) available for a reserve price of DDK 25 million each. In the 2.3 GHz band, there were two lots for 20 MHz available for a reserve price of DDK 25 million. In the 3.5 GHz band there were three categories of lots. First, there were three lots in the 3.5 GHz band available (3.5-D). The reserve price for such a lot was DDK 75 million. One such lot corresponds to 80 MHz in the 3.5 GHz spectrum and 400 MHz in the 26 GHz spectrum. Second, there was a single lot of 60 MHz (3.5-P) for a reserve price of DDK 25 million in the 3.5 GHz band with a leasing obligation. Third, there were nine 10 MHz lots (3.5-U) for a reserve price of DDK 25 million. The second CMRA was for the remaining lots in the 26 GHz band. In total there were 2850 MHz unpaired frequencies in the 26 GHz band. After subtracting the 1200 MHz sold in the first auction through the 3.5-D lots, there were 1650 MHz available (in lots of 200 MHz and 250 MHz) in the second CMRA. The reserve price was about DKK 5 million per lot.

Each of the three bidders won a 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0, two 2x5 MHz lots in the 2.1 GHz band, and a 3.5-D lot (80 MHz in the 3.5 GHz band and 400 MHz in the 26 GHz band).

In addition, bidder A won 40 MHz in the 3.5 GHz band (four 3.5-U lots) in the first CMRA and 600 MHz in the 26 GHz band in the second CMRA. Bidder A's total payment was DKK 540,525,000.

In addition to the above, bidder B won the 1.5-B lot, four 1.5-M lots, the two lots in the 2.3 GHz band, and 50 MHz in the 3.5 GHz band (five 3.5-U lots). In the second CMRA, bidder B won 850 MHz in the 26 GHz band. Bidder B's total payment was DKK 794,685,000.

In addition to the above, bidder C won the 1.5-T lot, four 1.5-M lots, and the 60 MHz in the 3.5 GHz spectrum with the leasing obligation. In the second CMRA, bidder C won 200 MHz in the 26 GHz band. Bidder C's total payment was DKK 740,976,000.

We first look at the differences between bidders B and C. Bidder B won the two 2.3-U lots in the 2.3 GHz band (40 MHz in total), 10 MHz less in the 3.5 GHz band (but without the leasing obligation), and 650 MHz more in the 26 GHz band. Bidder B paid DKK 53,709,000 more than bidder C. Bidder B's final assignment seems to dominate bidder C's and cost only DKK 54 million more. Compare this number to the reserve price of DKK 100 million for the 2.3 GHz band alone. Hence, we suspect that bidder B used additional bids to win the large package (as under CMRA-truthful bidding with decreasing marginal values).

Next, we compare the outcomes of bidders A and B. Bidder B paid DKK 254 million more than bidder A and got the additional 45 MHz in the 1500 MHz band, 40 MHz in the 2300 MHz band, 10 MHz in the 3.5 GHz band, and 250 MHz in the 26 GHz band. The reserve price for the additional lots won by bidder B is DKK 180 million. Hence, bidder B paid DKK 74 million in excess of the reserve price.

Comparing bidders A and C, bidder C won 45 MHz in the 1500 MHz band while bidder A did not win any lot in this category. Bidder C won 20 MHz more in the 3.5 GHz band (but subject to the leasing obligation), and 400 MHz less in the 26 GHz band. Bidder A paid DKK 200 million less, however. The reserve price of the 45 MHz in the

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
1.5-B	25 MHz in the 1500 MHz band (bottom)	1	10		1	
1.5-M	5 MHz in the 1500 MHz band	8	10		4	4
1.5-T	25 MHz in the 1500 MHz band (top)	1	10			1
2.1-U	2x5 MHz in the 2.1 GHz band	6	25	2	2	2
2.3-U	20 MHz in the 2.3 GHz band	2	50		2	
3.5-D	80 MHz in 3.5 GHz + 400 MHz in 26 GHz	3	75	1	1	1
3.5-P	60 MHz in 3.5 GHz (leasing obligation)	1	25			1
3.5-U	10 MHz in the 3.5 GHz band	9	25	4	5	
26-U	200 MHz/250 MHz in the 26 GHz band	8	5	3	4	1
Non-competitive						
2.1-D	2x10 MHz in the 2.1 GHz band	3	0	1	1	1
Expenditure						
	In million DKK		865	541	795	741

Table SA.2: Supply and auction outcome in the 2021 Danish spectrum auction
Note: R = reserve price in million DKK

1500 MHz band was DDK 50 million.

We conclude that it is likely that bidder B won with an additional bid. Due to the many prices, we cannot say whether bidders A and B won their headline demands or with additional bids. There is, however, no evidence for risk-free demand reduction.

SA.3 Illustration of CMRA-truthful bidding

We now illustrate how the CMRA progresses under CMRA-truthful bidding in the setting of Section 5; there are two bidders ($n = 2$) and the single good ($m = 1$) is perfectly divisible.

SA.3.1 Decreasing marginal values

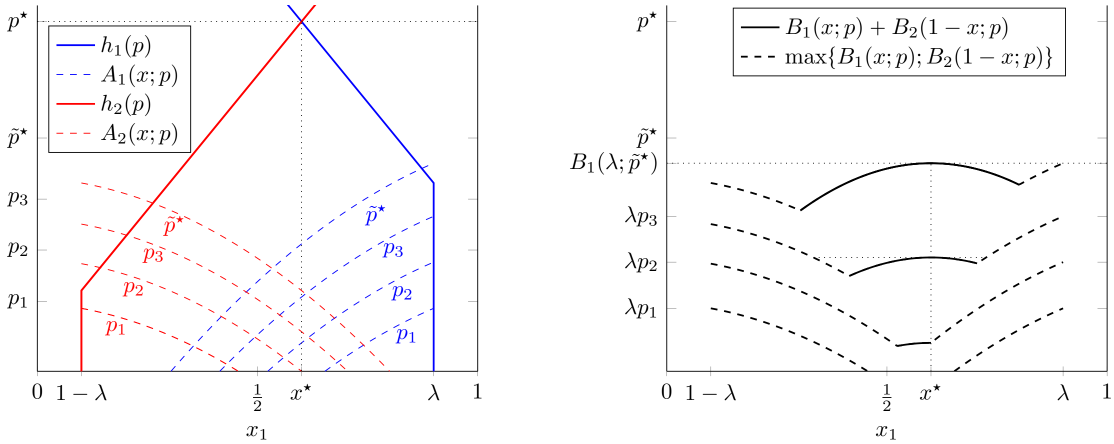
Figure SA.1 illustrates the results of Section 4 in the symmetric-caps case. Figure SA.1a shows the headline demands and additional bids at various clock prices. Bidder 1 is stronger, so the efficient share is $x_1^\star > \frac{1}{2}$ while bidder 2's efficient share is $x_2^\star < \frac{1}{2}$. Solid lines are headline demands $h_i(p)$ and dashed lines are truthful additional bids $A_i(x; p)$ as in Eq. (1) in the main text). Figure SA.1b depicts the respective revenue from feasible allocations. The solid line $B_1(x; p) + B_2(1 - x; p)$ shows revenue for allocations in which a bid of each bidder is accepted since this is required by the CMRA closing rule (recall that bids are $-\infty$ for shares that bidders do not bid on). The dashed line is $\max\{B_1(x; p), B_2(1 - x; p)\}$ for allocations that do not receive non-negative bids from both bidders: this is revenue that can be obtained by accepting only one bidder's bid.

Let us consider how the bids and allocations change as the clock price increases. As a benchmark, consider a simple clock auction (or a CMRA with clock-truthful bidding). We simply increase the clock price and follow the headline demands in solid lines in Figure SA.1a. The auction ends at clock price $p^\star$ with market clearing.

Under CMRA-truthful bidding, both bidders submit headline demands and additional bids. When the clock price p is low, only quantities close to λ receive additional bids.

- *Clock price p_1.* At this clock price, each bidder's headline demand is λ, yielding surplus $U_i(\lambda) - \lambda p_1$. Recall that the additional bids are given by Eq. (1) in the main text. At p_1, the additional bids range from λp_1 (for a quantity λ) to zero (for a smaller quantity that keeps the bidder indifferent). At p_1, there is no feasible allocation that receives bids from both bidders (the dashed lines do not intersect in Figure SA.1b). The auction continues as it is not possible to accept a bid by each bidder in the revenue-maximizing allocation.

- *Clock price p_2.* This is the lowest price at which both bidders bid on their respective efficient quantities. Bidder 1 submits a strictly positive additional bid on $x_1^\star$ at clock price p_2, while bidder 2 submits an additional bid of 0 on $x_2^\star$. From now on the efficient allocation can in principle be allocated as it receives bids from both bidders. Figure SA.1b reveals, however, that the efficient allocation is not revenue-maximizing. Bidder 1's headline demand is still λ, and allocating λ to bidder 1 raises a revenue of λp_2. Observe that bidder 2 bids less than λp_2 on λ because marginal values are decreasing and because λ is not the headline demand. As bidder 2 does



(a) Bidding functions

(b) Revenue of various allocations

Figure SA.1: CMRA-truthful bidding with decreasing marginal values.
Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

not bid on $1-\lambda$ at clock price p_2, the revenue-maximizing allocation features only bids by one bidder and the auction continues.

- *Clock price* p_3. Both bidders have now raised their additional bids on their respective efficient share. As we can see in Figure SA.1b, the efficient allocation $x^\star$ locally maximizes revenue. However, $x^\star$ does not yield a global revenue maximum as bidder 1's bid on λ leads to a higher revenue of λp_3.

- *Clock price* $\tilde{p}^\star$. The additional bids are now sufficiently high so that the efficient allocation $x^\star$ is revenue-maximizing. The auction ends at $\tilde{p}^\star$ at which

$$B_1(x_1^\star;\tilde{p}^\star) + B_2(x_2^\star;\tilde{p}^\star) = \max_i B_i(\lambda;\tilde{p}^\star). \tag{7}$$

  Note that at price $\tilde{p}^\star$ bidder 2 does not yet bid on $1-\lambda$, so $(\lambda, 1-\lambda)$ is not a feasible allocation. Revenue is lower than $\tilde{p}^\star$ and lower than $p^\star$ (Fig. SA.1b).

SA.3.2 Non-decreasing marginal values

We now consider non-decreasing marginal values. Since no competitive equilibrium needs to exist, we prove a revenue comparison between clock-truthful and CMRA-truthful bidding.

Proposition SA.1. *Let the marginal values be non-decreasing.*

(i) Clock-truthful bidding in the CMRA leads to excess supply. The clock ends at clock price $p = \min_i U_i(\lambda_i)/\lambda_i$. *The final auction allocation is inefficient.*

(ii) *If $\lambda_1 = \lambda_2$, then ex-post revenue under CMRA-truthful bidding is lower than under clock-truthful bidding; if $\lambda_1 > \lambda_2$, then the ex-post revenue comparison is ambiguous.*

Proof. The clock-truthful demand is λ_i until the price reaches $U_i(\lambda_i)/\lambda_i$. For higher prices demand equals 0. Hence, the clock ends at price $\min U_i(\lambda_i)/\lambda_i$ and it does so with excess supply. Due to positive marginal values for all shares below λ_i, the efficient allocation does not feature excess supply. The outcome is inefficient.

Let $\lambda_1 = \lambda_2 = \lambda$. Under clock-truthful bidding, the clock ends at clock price $\min_i U_i(\lambda)/\lambda$. Under CMRA-truthful bidding the auction ends at a lower clock price, namely $\min_i p_i^f$. As ex-post revenue is λ times the final clock price in both cases, revenue is lower under CMRA-truthful bidding due to the lower final clock price.

The following numerical example proves that revenue can also be higher under CMRA-truthful bidding. Let $\lambda_1 = \frac{7}{8}$, $\lambda_2 = \frac{6}{8}$, $U_1(\lambda_1) = 21$, $U_1(1-\lambda_2) = 1$, $U_2(\lambda_2) = \frac{39}{2}$, and $U_2(1-\lambda_1) = \frac{1}{2}$.[1] The allocation $(\lambda_1, 1-\lambda_1)$ is efficient. Under clock-truthful bidding, the CMRA ends at clock price $\min_i U_i(\lambda_i)/\lambda_i = U_1(\lambda_1)/\lambda_1 = 24 < 26 = U_2(\lambda_2)/\lambda_2$. Consider CMRA-truthful bidding. Since $(\lambda_1, 1-\lambda_1)$ is efficient, Theorem 1 implies that the CMRA must end at clock price $p_2^f = 76/3$. Revenue is $p_2^f \lambda_1 = \frac{133}{6}$, which is more than the revenue under clock-truthful bidding: $\lambda_2 U_1(\lambda_1)/\lambda_1 = 18$. $\square$

Figure SA.2 illustrates Proposition SA.1 for symmetric caps. As before, Figure SA.2a shows the headline demands and the additional bids, while Figure SA.2b shows revenue under different allocations.

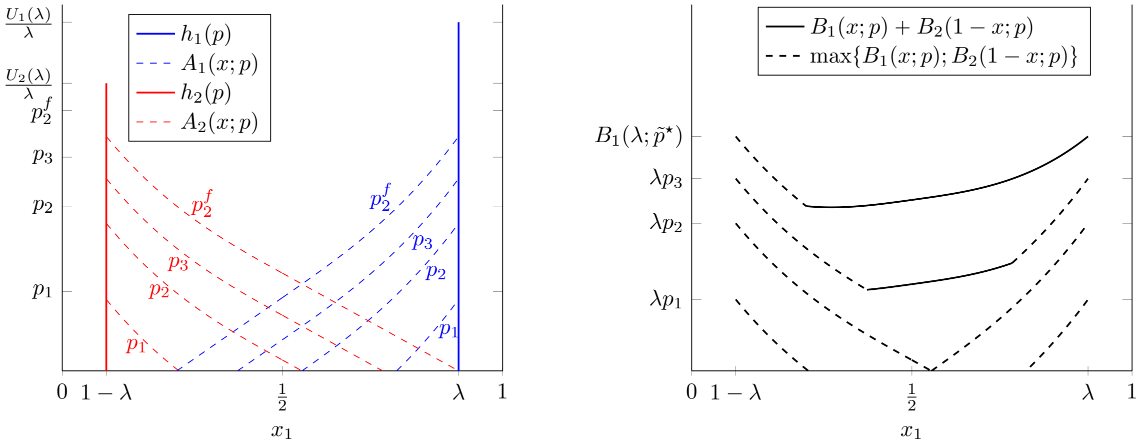
Once again, let us first consider the outcome of a clock auction or of clock-truthful bidding in the CMRA. Figure SA.2a shows that, due to increasing marginal values, bidder i's clock-truthful headline demand is λ for $p \leqslant U_i(\lambda)/\lambda$ and 0 for higher prices. Hence, the auction ends at clock price $p = \min_i U_i(\lambda)/\lambda = U_2(\lambda)/\lambda$. As bidder 2 drops demand to 0 at price $U_2(\lambda)/\lambda$, the auction ends with excess supply of $1-\lambda$. Bidder 1 wins quantity λ and the revenue is $U_2(\lambda)$. Note that in this case, clock-truthful bidding in the CMRA is equivalent to a VCG auction restricted to selling λ as a bundle (i.e., a second-price auction for λ), so the revenue is lower than in the (unrestricted) VCG auction.[2]

Under CMRA-truthful bidding, bidders' headline demands are as under clock-truthful bidding but they also submit additional bids. For low clock prices, bidders submit few additional bids, but as the clock price rises, bidders increase their additional bids both on the intensive and extensive margins.

- *Clock price p_1.* There is no feasible allocation that receives non-negative bids from both bidders. Hence, the auction continues.

[1] It is straightforward to check that the utility functions are consistent with non-decreasing marginal values. The difference $\lambda_1 - (1-\lambda_2) = \lambda_2 - (1-\lambda_1) = \frac{5}{8}$ and $U_1(\lambda_1) - U_1(1-\lambda_2) = 20 > 19 = U_2(\lambda_2) - U_2(1-\lambda_1)$.

[2] We are grateful to an anonymous referee for this observation.



(a) Bidding functions

(b) Revenue of various allocations

Figure SA.2: CMRA-truthful bidding with non-decreasing marginal values

Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

- *Clock price p_2.* There are now feasible allocations that receive non-negative bids from both bidders. These allocations are not revenue-maximizing as $B_i(\lambda; p) = \lambda p$ yields higher revenue.

- *Clock price p_3.* Feasible allocations that receive non-negative bids from both bidders are still not revenue-maximizing. Bidder 1's marginal values (and additional bids) are higher and non-decreasing, so allocating more to bidder 1 increases revenue.

- *Clock price p_2^f.* At this price, the weaker bidder 2 places an additional bid of 0 on $1-\lambda$. More generally, there is a *final price* p_i^f at which bidder i bids 0 on $1-\lambda$ as this bidder is indifferent between winning λ for a payment of $p_i^f\lambda$ and winning $1-\lambda$ for free. The indifference condition $U_i(\lambda) - p_i^f\lambda = U_i(1-\lambda)$ transforms to

$$p_i^f = \frac{U_i(\lambda_i) - U_i(1-\lambda_j)}{\lambda_i}.$$

  With bidder 2's additional bid, it is now possible to accept a bid by each bidder in the revenue-maximizing allocation $(\lambda, 1-\lambda)$ (Figure SA.2b). Therefore, the CMRA ends in market-clearing. The revenue is λp_2^f, which is lower than $\lambda U_2(\lambda)/\lambda$.