

Corrigendum to: Andersson, Tommy, Ágnes Cseh, Lars Ehlers, and Albin Erlanson. 2021. “Organizing Time Exchanges: Lessons from Matching Markets.” *American Economic Journal: Microeconomics* 13(1): 338–373

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We have been informed that the parts of the proof of Theorem 1 that are presented in “Appendix B.2: Replacement Result” in our article contains a critical error.¹ While the theorem remains correct, the original proof was flawed. The problem in the original proof is that non-manipulability is proved for a priority mechanism that is defined over the individual rational set, instead of the set of individual rational and maximal allocations. Correcting the proof amounts to allowing the mechanism to select from the set of individual rational and maximal matchings. This can be achieved by adding a “large enough” constant to the inner arc weights. The main structure and result of the paper are unaffected by this correction. The correction only applies to the supporting argument provided in Appendix B.2.

The updated and correct proof is provided below and should be seen as a full replacement of the original Appendix B.2.

Revised Appendix B.2: Replacement Result

This section demonstrates that by placing appropriate weights on the arcs in the graph D , the maximum weight circulations correspond to the outcome of the min-cost flow problem used in Section IV to identify the outcome of the priority mechanism (Proposition 2). This result implies that the circulation-based model can be adopted in the proof of Theorem 1. We remark that both maximum weight circulations and min-cost flows can be computed efficiently, or more precisely, in $O(|E|^2 \log |V|)$ time in a graph with $|V|$ vertices and $|E|$ arcs, which is strongly polynomial time (Orlin, 1993).

Let π be a priority-ordering. Let t_{max} be the largest endowment of any agent in N , note that $t_{max} > 1$.² We define the weight $w(u, v)$ on each arc (u, v) in the directed graph $D = (V, A)$ by:

$$w(u, v) = \begin{cases} n \cdot t_{max}^{2n+1} + t_{max}^{2(n+1-\pi(i))} & \text{if } (u, v) = (i^{in}, i^{out}), \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

We now illustrate our transformation from the priority mechanism to the circulation-based model by means of a simple example.

Example 4: Agents are denoted by i, j, k and l . The upper bounds on the acceptable goods are as follows: $\bar{t}_{ki} = 1, \bar{t}_{ji} = 2, \bar{t}_{kj} = 2, \bar{t}_{lj} = 1, \bar{t}_{ik} = 3, \bar{t}_{il} = 1, \bar{t}_{jl} = 3$. All other upper bounds are set to 0. Each agent has an endowment of 3, and their priority-order is alphabetic. In the circulation-based model, there are two vertex copies to each agent, connected by an inner arc. The number of goods that each agent is willing to accept from another agent translates into an upper capacity on the regular arc connecting the out-vertex of the provider and the in-vertex of the receiver. In Figure 1,

¹We are grateful to Alexander Westkamp and Vikram Manjunath for identifying the error in the original proof.

²The model and our results are only novel for $t_{max} > 1$, otherwise our model coincides with the standard house allocation where all agents have an endowment of one house each.

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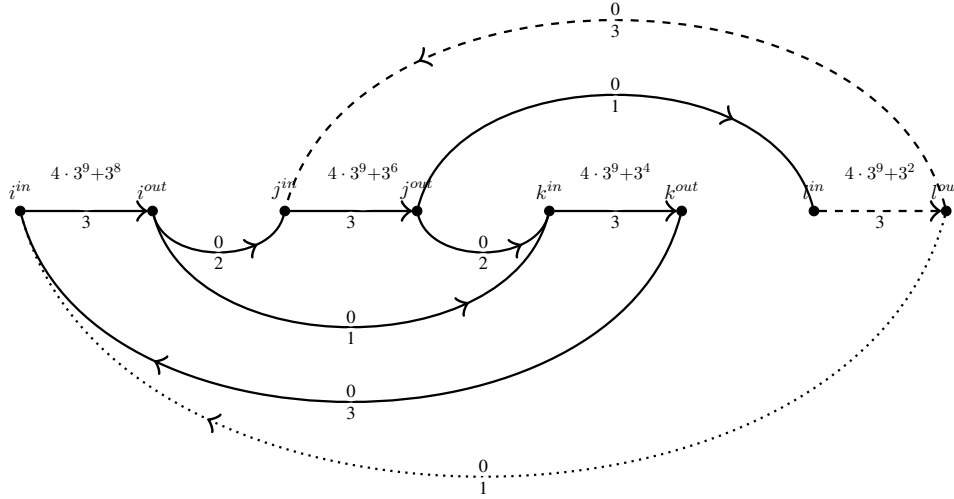


Figure 1: The circulation-based model for the instance in Example 4.

Let $w(C)$ denote the *weighted sum of flow values* of the agents in N at circulation C , i.e., $w(C) = \sum_{i \in N} C(i^{in}, i^{out}) \cdot w(i^{in}, i^{out})$.

PROPOSITION 2: *For any given profile $R \in \bar{\mathcal{R}}$, let C be a maximum weight circulation where the weights are defined by condition (9). Let C' be the circulation corresponding to an allocation x' selected for R by a priority mechanism φ based on π . Then $C'(i^{in}, i^{out}) = C(i^{in}, i^{out})$ for each $i \in N$.*

Proof. As in the statement of the proposition, let C be a maximum weight circulation and let C' be the circulation corresponding to an allocation x' selected by a priority mechanism.

We first prove that C corresponds to a maximal allocation. We have:

$$\begin{aligned}
w(C) &= \sum_{i \in N} C(i^{in}, i^{out}) \cdot w(i^{in}, i^{out}) = \sum_{i \in N} C(i^{in}, i^{out}) \cdot (n \cdot t_{max}^{2n+1} + t_{max}^{2(n+1-\pi(i))}) = \\
&= n \cdot t_{max}^{2n+1} \sum_{i \in N} C(i^{in}, i^{out}) + \sum_{i \in N} C(i^{in}, i^{out}) \cdot t_{max}^{2(n+1-\pi(i))} \leq \\
&= n \cdot t_{max}^{2n+1} \sum_{i \in N} C(i^{in}, i^{out}) + \sum_{i \in N} C(i^{in}, i^{out}) \cdot t_{max}^{2n} \leq \\
&= n \cdot t_{max}^{2n+1} \sum_{i \in N} C(i^{in}, i^{out}) + n \cdot t_{max} \cdot t_{max}^{2n} = \\
&= n \cdot t_{max}^{2n+1} \left(\sum_{i \in N} C(i^{in}, i^{out}) + 1 \right),
\end{aligned}$$

where the first inequality follows from $\pi(i) \geq 1$ for all $i \in N$ and the second inequality follows from $C(i^{in}, i^{out}) \leq t_{max}$ for all $i \in N$. Suppose that C is a maximum weight circulation but not a maximal allocation. Then there exists a feasible circulation $\tilde{C}$ with a higher total sum of flow values than C , i.e. $\sum_{i \in N} \tilde{C}(i^{in}, i^{out}) \geq \sum_{i \in N} C(i^{in}, i^{out}) + 1$ and we have that:

$$w(\tilde{C}) \geq n \cdot t_{max}^{2n+1} \left[\sum_{i \in N} C(i^{in}, i^{out}) + 1 \right] + \sum_{i \in N} \tilde{C}(i^{in}, i^{out}) t_{max}^{2(n+1-\pi(i))} > w(C)$$

where the strict inequality follows from the steps above and $|N| \geq 1$ and the right sum of the term in between the inequalities is greater than zero. Now this is contradiction as C is not a maximum weight circulation. Further, as the priority mechanism selects a maximal allocation x' by definition. From this point on we can restrict our attention to maximal allocations and their corresponding circulation representations.

Suppose, to obtain a contradiction, that $C'(j^{in}, j^{out}) \neq C(j^{in}, j^{out})$ for some $j \in N$. Let agent i be the agent with the highest priority in π where this holds. Suppose also, without loss of generality, that $\pi(k) = k$ for all $k \in N$. To reach the desired contradiction, we consider two cases.

Case (i): $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$. In this case, the maximum weight circulation C assigns a higher allocation value to agent i than the priority mechanism. We show by induction that this contradicts the rules of the priority mechanism. Suppose first that agent i is the highest-ranked agent according to the priority-order π and recall that the priority mechanism, by construction, restricts the set of maximal allocations to those that maximize the allocation value of i (see condition (3) in Section III). Thus, there is no maximal allocation that assigns agent i a higher allocation value than the allocations in this chosen set, and, consequently, no maximum weight circulation that assigns agent i a higher value. Hence, agent i cannot be the agent with the highest priority and, $C'(i^{in}, i^{out}) = C(i^{in}, i^{out})$. Suppose now that agent i is the second-highest-ranked agent according to the priority-order π . Again, by condition (3) this agent restricts the set of allocations further, from the set of maximal allocations where the total flow is maximized for the agent with the highest priority. Thus, the maximum weight circulation C is still in the chosen set when agent i restricts the set of allocations further, and it can, consequently, not have a higher allocation value for agent i than C' . This argument can be repeated inductively to reach the conclusion that it cannot be the case that $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$.

Case (ii): $C'(i^{in}, i^{out}) > C(i^{in}, i^{out})$. Note first that both C and C' are feasible circulations at profile R . As maximum circulations necessarily translate to maximal allocations, both C and C' corre-

spend to a maximal allocation, and thus, the weight components derived from the weight component $n \cdot t_{max}^{2n+1} \sum_{i \in N} C(i^{in}, i^{out})$ are equal in $w(C)$ and $w(C')$.

Because agent i is the agent with the highest priority in π where $C'(i^{in}, i^{out}) \neq C(i^{in}, i^{out})$, by assumption, it follows that $C'(k^{in}, k^{out}) = C(k^{in}, k^{out})$ for all agents $k = 1, \dots, i - 1$. It will be demonstrated that agents $i + 1, \dots, n$ cannot make up for the loss C suffered on the inner arc (i^{in}, i^{out}) and thus C cannot be of maximum weight since C' is a feasible circulation at profile R . Recall first that the set $\mathbb{N}_0$ contains only positive integers, so the difference between $C'(i^{in}, i^{out})$ and $C(i^{in}, i^{out})$ is at least 1. By construction of the weights on the inner arcs, defined by condition (9), it then follows that:

$$[C'(i^{in}, i^{out}) - C(i^{in}, i^{out})] \cdot t_{max}^{2(n-i+1)} \geq t_{max}^{2(n-i+1)} = \frac{t_{max}^{2n+1}}{t_{max}^{2i}}. \quad (10)$$

Note next that, in the extreme case, all agents with lower priorities than agent i have flow value zero in C' and a flow value of t_{max} in C . This means that the weighted sum of the flow values at agents $i + 1, \dots, n$ at circulation C is at most:

$$t_{max} \cdot \sum_{j=i+1}^n t_{max}^{2(n-j+1)} = t_{max} \left[\frac{t_{max}^{2n+1}}{t_{max}^{2(i+1)}} + \frac{t_{max}^{2n+1}}{t_{max}^{2(i+2)}} + \dots + \frac{t_{max}^{2n+1}}{t_{max}^{2n}} \right]. \quad (11)$$

Now, the value of the sum (11) is strictly lower than the right hand side of inequality (10) as

$$\sum_{j=i+1}^n \frac{1}{t_{max}^{j-i}} \leq \sum_{k=1}^n \frac{1}{2^k} < 1$$

by $t_{max} \geq 2$. Consequently, even in the extreme case when all agents with lower priorities than agent i have flow value zero in C' and a flow value of t_{max} in C , it holds that $w(C') > w(C)$. However, this contradicts that C is a maximum weight circulation since C' is a feasible circulation at graph D_R . $\square$

References

Orlin, James B. 1993. "A Faster Strongly Polynomial Minimum Cost Flow Algorithm." *Operations Research* 41(2): 338–350.