

Parallel Digital Currencies and Sticky Prices

ONLINE APPENDIX

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MATHEMATICAL DERIVATIONS FOR THE LINEARIZED NK MODEL

B1. Households

The lifetime utility of a representative household is given by:

$$(B1) \quad E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, L_t, N_t) Z_t$$

$$(B2) \quad u(C_t, L_t, N_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \frac{L_t^{1-\xi} - 1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

The marginal utility of consumption, liquidity, and labor:

$$(B3) \quad u_{C,t} = C_t^{-\sigma}$$

$$(B4) \quad u_{L,t} = L_t^{-\xi}$$

$$(B5) \quad u_{N,t} = -N_t^{\varphi}.$$

The household's budget constraint is

$$(B6) \quad C_t + B_t + \sum_{j=1}^J L_{j,t} = \frac{\exp(i_{t-1}) B_{t-1}}{\Pi_t} + \sum_{j=1}^J \frac{L_{j,t-1}}{\Pi_t} \frac{\mathcal{E}_{j,t}}{\mathcal{E}_{j,t-1}} + W_t N_t + \Gamma_t$$

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The first-order conditions with respect to N_t , B_t , and $L_{j,t}$, are

$$(B7) \quad N_t : W_t = \frac{N_t^\varphi}{C_t^{-\sigma}}$$

$$(B8) \quad B_t : C_t^{-\sigma} = \beta \exp(i_t) E_t \left[C_{t+1}^{-\sigma} \frac{1}{\Pi_{t+1}} \frac{Z_{t+1}}{Z_t} \right]$$

$$(B9) \quad L_{j,t} : \frac{L_t^{-\xi}}{C_t^{-\sigma}} = 1 - \beta E_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1}{\Pi_{t+1}} \frac{\mathcal{E}_{j,t+1}}{\mathcal{E}_{j,t}} \frac{Z_{t+1}}{Z_t} \right]$$

Upon log-linearization at the first order around the zero-inflation steady state, the optimality conditions are expressed as:

$$(B10) \quad w_t = \varphi n_t + \sigma c_t$$

$$(B11) \quad c_t = E_t[c_{t+1}] - \frac{1}{\sigma} \left(\hat{i}_t - E_t[\pi_{t+1}] \right) + \frac{1}{\sigma} (1 - \rho_z) z_t$$

$$(B12) \quad -\sigma c_t = \frac{\beta}{1 - \beta} [\sigma (c_t - E_t[c_{t+1}]) - E_t[\pi_{t+1}] + E_t[\Delta e_{j,t+1}] - (1 - \rho_z) z_t]$$

B2. Firms

The production function of firm i that prices in currency j is

$$(B13) \quad Y_{j,t}(i) = A_t N_{j,t}(i)^{1-\alpha}$$

Recall that W_t denotes the real wage. The real marginal cost of a firm in sector j is then given by

$$(B14) \quad MC_t(i) = \frac{W_t}{(1 - \alpha) A_t N_{j,t}(i)^{-\alpha}}$$

$$(B15) \quad = \frac{W_t}{(1 - \alpha) A_t} \left(\frac{Y_{j,t}(i)}{A_t} \right)^{\frac{\alpha}{1-\alpha}}$$

$$(B16) \quad = \frac{W_t}{(1 - \alpha) A_t^{\frac{1}{1-\alpha}}} Y_t^{\frac{\alpha}{1-\alpha}} \left(\frac{\mathcal{E}_{j,t} P_t(i)}{P_t} \right)^{-\frac{\alpha \eta}{1-\alpha}}$$

where we have used Eq. (10) as well as $C_t = Y_t$ and $C_t(i) = Y_t(i)$ in the last step. The average real marginal cost of the economy

$$(B17) \quad MC_t = \sum_{j=1}^J \int_{V_j(t)} MC_t(i) di$$

$$(B18) \quad = \frac{W_t}{(1-\alpha) A_t^{\frac{1}{1-\alpha}}} Y_t^{\frac{\alpha}{1-\alpha}} \left[\sum_{j=1}^J \int_{V_{j,t}} \left(\frac{\mathcal{E}_{j,t} P_t(i)}{P_t} \right)^{-\frac{\alpha \eta}{1-\alpha}} di \right]$$

In order to log-linearize this equation when $V_{j,t} \equiv V_j$ is independent of time, examine first the term

$$Q_t = \sum_{j=1}^J \int_{V_j} \left(\frac{\mathcal{E}_{j,t} P_t(i)}{P_t} \right)^{-\frac{\alpha \eta}{1-\alpha}} di$$

Log-linearization delivers

$$(B19) \quad -\frac{1-\alpha}{\alpha \eta} \bar{Q} q_t + \left(\sum_{j=1}^J \int_{V_j} \left(\frac{\bar{\mathcal{E}}_j \bar{P}(i)}{\bar{P}} \right)^{-\frac{\alpha \eta}{1-\alpha}} \right) p_t = \sum_{j=1}^J \int_{V_j} \left(\frac{\bar{\mathcal{E}}_j \bar{P}(i)}{\bar{P}} \right)^{-\frac{\alpha \eta}{1-\alpha}} (e_{j,t} + p_t(i)) di$$

Likewise, log-linearizing the definition of the price index delivers

$$(B20) \quad p_t = \sum_{j=1}^J \int_{V_j} \left(\frac{\bar{\mathcal{E}}_j \bar{P}(i)}{\bar{P}} \right)^{1-\eta} (e_{j,t} + p_t(i)) di$$

Note now that

$$\frac{\bar{\mathcal{E}}_j \bar{P}(i)}{\bar{P}} = 1$$

as all firms will choose the same price expressed in dollars at the steady state, and that price therefore equals the general price index. Use this and (B20) in (B19) to find that $q_t = 0$. The log-linearized real marginal cost is therefore

$$(B21) \quad mc_t = w_t + \frac{\alpha}{1-\alpha} y_t - \frac{1}{1-\alpha} a_t$$

It follows that an individual firm's real marginal cost and the average real marginal cost, after log-linearization, are associated with the following equation:

$$(B22) \quad mc_t(i) = mc_t - \frac{\alpha \eta}{1-\alpha} [p_t(i) + e_{j,t} - p_t]$$

The firm seeks to maximize real profits per its choice of the price $P_t(i)$. With flexible prices, the desired price $P_t(i) = \tilde{P}_{j,t}$ is chosen so as to equalize real marginal revenue to real marginal cost, taking into account the demand function (10) with $C_t = Y_t$ and $C_t(i) = Y_t(i)$. This leads to the markup equation

$$(B23) \quad \frac{\eta - 1}{\eta} \frac{\mathcal{E}_{j,t} \tilde{P}_{j,t}}{P_t} = MC_t(i)$$

where $MC_t(i)$ is evaluated at the price $P_t(i) = \tilde{P}_{j,t}$. The log-linearized desired price is

$$(B24) \quad \tilde{p}_{j,t} = mc_{j,t|t} + p_t - e_{j,t}$$

where (from [Eq. \(B22\)](#))

$$(B25) \quad mc_{j,t|t} = mc_t - \frac{\alpha \eta}{1 - \alpha} [\tilde{p}_{j,t} + e_{j,t} - p_t]$$

The desired price can be solved as

$$(B26) \quad \tilde{p}_{j,t} = \Theta mc_t + p_t - e_{j,t}$$

where

$$(B27) \quad \Theta = \frac{1 - \alpha + \alpha \eta}{1 - \alpha}$$

Note that the dollar-denominated desired price, $\tilde{p}_{j,t} + e_{j,t}$, is independent of currency choice. When prices are sticky, the log-linearized optimal price is

$$(B28) \quad p_{j,t}^* = (1 - \beta \theta_j) \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell E_t [mc_{t+\ell|t} + p_{t+\ell} - e_{j,t+\ell}]$$

To simplify the equation, use [Eq. \(B22\)](#) to establish the relationship between the optimizing firm's real marginal cost with the economy's average marginal cost:

$$(B29) \quad mc_{t+\ell|t} = mc_{t+\ell} - \frac{\alpha \eta}{1 - \alpha} [p_{j,t}^* + e_{j,t+\ell} - p_{t+\ell}]$$

Substitute into [Eq. \(B28\)](#) and simplify:

(B30)

$$p_{j,t}^* = (1 - \beta \theta_j) \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t \left[\text{mc}_{t+\ell} - \frac{\alpha \eta}{1 - \alpha} [p_{j,t}^* + e_{j,t+\ell} - p_{t+\ell}] + p_{t+\ell} - e_{j,t+\ell} \right]$$

(B31)

$$= \Theta^{-1} (1 - \beta \theta_j) \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t [\Theta \text{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell} - (1 - \Theta) p_{j,t}^*]$$

(B32)

$$= -(\Theta^{-1} - 1) p_{j,t}^* + \Theta^{-1} (1 - \beta \theta_j) \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t [\Theta \text{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell}]$$

(B33)

$$= (1 - \beta \theta_j) \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t [\tilde{p}_{j,t+\ell}]$$

which can be written in a recursive form

$$(B34) \quad p_{j,t}^* = \beta \theta_j \mathbb{E}_t [p_{j,t+1}^*] + (1 - \beta \theta_j) \tilde{p}_{j,t}$$

The law of motion for inflation is derived using the sectoral price index

$$(B35) \quad p_{j,t}^* - p_{j,t-1} = \beta \theta_j \mathbb{E}_t [p_{j,t+1}^* - p_{j,t}] + (1 - \beta \theta_j) \tilde{p}_{j,t} - p_{j,t-1} + \beta \theta_j p_{j,t}$$

$$(B36) \quad = \beta \theta_j \mathbb{E}_t [p_{j,t+1}^* - p_{j,t}] - (1 - \beta \theta_j) (p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t}$$

From the identity $p_{j,t} = \theta p_{j,t-1} + (1 - \theta) p_{j,t}^*$, we have $\pi_{j,t} = (1 - \theta) (p_{j,t}^* - p_{j,t-1})$. The above equation becomes

$$(B37) \quad (1 - \theta)^{-1} \pi_{j,t} = (1 - \theta)^{-1} \beta \theta_j \mathbb{E}_t [\pi_{j,t+1}] - (1 - \beta \theta_j) (p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t}$$

Rearrange terms to obtain the law of motion for sectoral inflation

$$(B38) \quad \pi_{j,t} = \beta \mathbb{E}_t [\pi_{j,t+1}] - \lambda_j (p_{j,t} - \tilde{p}_{j,t})$$

where $p_{j,t} - \tilde{p}_{j,t}$ is interpreted as the price markup.

B3. *Equilibrium*

The market clearing conditions for the goods and labor markets are

$$(B39) \quad Y_t = C_t$$

where the aggregate output is defined as

$$(B40) \quad Y_t \equiv \left(\sum_{j=1}^J \int_{V_j} Y_{j,t}(i)^{1-\frac{1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}$$

The labor market clears when

$$(B41) \quad N_t = \sum_{j=1}^J \int_{V_j} N_{j,t}(i) di$$

$$(B42) \quad = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} D_t$$

where $D_t \equiv \left[\sum_{j=1}^J \int_v \left(\frac{\varepsilon_{j,t} P_t(i)}{P_t} \right)^{-\frac{\eta}{1-\alpha}} di \right]$ is a version of price dispersion for a multi-sector economy. The price dispersion is elaborated as

$$(B43) \quad D_t = \sum_{j=1}^J \left(\frac{\varepsilon_{j,t} P_{j,t}}{P_t} \right)^{-\frac{\eta}{1-\alpha}} \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B44) \quad = \sum_{j=1}^J \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B45) \quad = \sum_{j=1}^J \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} D_{j,t}$$

where $D_{j,t}$ is the sectoral price dispersion

$$(B46) \quad D_{j,t} = \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B47) \quad = v_j (1 - \theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B48) \quad = v_j (1 - \theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}}{P_{j,t}} \frac{P_{j,t-1}(i)}{P_{j,t-1}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B49) \quad = v_j (1 - \theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \Pi_{j,t}^{\frac{\eta}{1-\alpha}} \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t-1}} \right)^{-\frac{\eta}{1-\alpha}} di$$

$$(B50) \quad = v_j (1 - \theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \theta_j \Pi_{j,t}^{\frac{\eta}{1-\alpha}} D_{j,t-1}$$

The labor market condition is linearized as

$$(B51) \quad n_t = \frac{y_t - a_t}{1 - \alpha}$$

Note that the price dispersion vanishes at the first order.

B4. Deriving the Sectoral NKPC

Combining (B10), (B21), (B51) and $c_t = y_t$ yields

$$(B52) \quad mc_t = \left(\sigma + \frac{\alpha + \phi}{1 - \alpha} \right) y_t - \frac{\phi + 1}{1 - \alpha} a_t$$

With that, (B26) delivers

$$(B53) \quad p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left[\left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) y_t - \frac{\varphi + 1}{1 - \alpha} a_t \right]$$

The last equation eliminates the real wage using the household's optimality condition for the labor supply. Under flexible prices, $p_{j,t} = \tilde{p}_{j,t}$, $\hat{p}_{j,t} = 0$, and we have

$$(B54) \quad 0 = -\Theta \left[\left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) y_t^n - \frac{\varphi + 1}{1 - \alpha} a_t \right]$$

Solve Eq. (B54) for the natural level of output

$$(B55) \quad y_t^n = \psi_{ya} a_t$$

Take the difference between Eqs. (B53) and (B54). The price markup is expressed in terms of the output gap defined as $\tilde{y}_t \equiv y_t - y_t^n$:

$$(B56) \quad p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \tilde{y}_t$$

Substitute the price markup back into the law of motion for sectoral inflation. The sectoral NKPC is derived as:

$$(B57) \quad \pi_{j,t} = \beta E_t [\pi_{j,t+1}] + \kappa_j \tilde{y}_t - \lambda_j \hat{p}_{j,t}$$

B5. Dynamic IS Curve

Using the market clearing condition $y_t = c_t$, one can rewrite the optimality condition for government bonds as

$$(B58) \quad y_t = E_t[y_{t+1}] - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}]) + \frac{1}{\sigma} (1 - \rho_z) z_t$$

Subtracting the flexible counterpart from B58 gives

$$(B59) \quad \tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}] - r_t^n)$$

where the natural rate of interest is a linear combination of exogenous shocks

$$(B60) \quad r_t^n \equiv -\sigma (1 - \rho_a) \psi_{ya} a_t + (1 - \rho_z) z_t$$

B6. NKPC in Non-Linear Form

The firm's profit maximisation problem is expressed as

$$(B61) \quad \max_{P_{j,t}^*} \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[Q_{t,t+\ell} \left[\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} - \Psi_{t+\ell}(Y_{j,t+\ell|t}) \right] \right]$$

subject to the demand function

$$(B62) \quad Y_{j,t+\ell|t} = \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell}$$

where $\Psi_{t+\ell}(\cdot)$ is the real total cost of production. The first-order condition for price setting is

$$(B63) \quad \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[Q_{t,t+\ell} \mathcal{E}_{j,t+\ell} (1/P_{t+\ell}) \left(Y_{j,t+\ell|t} + P_{j,t}^* \frac{\partial Y_{j,t+\ell|t}}{\partial P_{j,t}^*} - \frac{P_{t+\ell} MC_{j,t+\ell|t}}{\mathcal{E}_{j,t+\ell}} \frac{\partial Y_{j,t+\ell|t}}{\partial P_{j,t}^*} \right) \right] = 0$$

which can be simplified to the following condition

$$(B64) \quad \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t} \right)^{-\eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} - \frac{\eta}{\eta-1} \frac{P_{j,t+\ell}}{P_{j,t}} MC_{j,t+\ell|t} \right) \right] = 0$$

Multiply both sides by $\hat{P}_{j,t}^\eta$

$$(B65) \quad \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \Pi_{j,t}^{*- \eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} - \frac{\eta}{\eta-1} \frac{P_{j,t+\ell}}{P_{j,t}} MC_{j,t+\ell|t} \right) \right] = 0$$

where $\Pi_{j,t}^* \equiv \frac{\mathcal{E}_t P_{j,t}^*}{P_{j,t}}$. Rearrange terms,

$$(B66) \quad \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \Pi_{j,t}^{*- \eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right]$$

$$(B67) \quad = \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \Pi_{j,t}^{*- \eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{\eta}{\eta-1} MC_{j,t+\ell|t} \right]$$

Substitute the expression for idiosyncratic marginal cost:

$$(B68) \quad \Pi_{j,t}^{* 1 + \frac{\alpha\eta}{1-\alpha}} \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} Y_{t+\ell} \right]$$

$$(B69) \quad = \frac{\eta}{\eta-1} \sum_{\ell=0}^{\infty} \theta_j^\ell E_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} \left(\frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right)^{-\frac{\alpha\eta}{1-\alpha}} Y_{t+\ell} MC_{t+\ell} \right]$$

which can be rewritten as

$$(B70) \quad \Pi_{j,t}^{* 1 + \frac{\alpha\eta}{1-\alpha}} x_{2,t} = \frac{\eta}{\eta-1} x_{1,t}$$

where

$$(B71) \quad x_{2,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t} + \beta \theta_j E_t \left[\Pi_{j,t+1}^{\eta-1} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\eta} x_{2,t+1} \right]$$

and

$$(B72) \quad x_{1,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t}^{-\frac{\alpha\eta}{1-\alpha}} MC_t + \beta \theta_j E_t \left[\Pi_{j,t+1}^{\frac{\eta}{1-\alpha}} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\eta} x_{1,t+1} \right]$$

The relative price follows

$$(B73) \quad \hat{P}_{j,t} = \hat{P}_{j,t-1} \frac{\Pi_t^e \Pi_{j,t}}{\Pi_t}$$

In the case of flexible prices, the desired price is given by

$$(B74) \quad \tilde{P}_{j,t} = \frac{P_t}{\mathcal{E}_{j,t}} \left(\frac{\eta}{\eta-1} MC_t \right)^{\Theta}$$

B7. Firm's Lifetime Profit in Non-Linear Form

Iterate the value function:

$$(B75) \quad \begin{aligned} \mathcal{V}_{j,t|t} &= E_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \Xi_{j,t+\ell|t} \right] \\ &= E_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \right] - E_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} MC_{j,t+\ell|t} Y_{j,t+\ell|t} \right] \end{aligned}$$

The value function of the firm is therefore

$$(B76) \quad \mathcal{V}_{j,t} = \mathcal{R}_{j,t} - \mathcal{C}_{j,t}$$

The revenue is given by

$$\begin{aligned}
 (B77) \quad \mathcal{R}_{j,t} &= \sum_{\ell=0}^{\infty} \theta_j^\ell Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \\
 (B78) \quad &= \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \\
 (B79) \quad &= \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \\
 (B80) \quad &= \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \right)^{1-\eta} Y_{t+\ell} \\
 (B81) \quad &= \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{1-\eta} Y_{t+\ell} \\
 (B82) \quad &= \frac{1}{C_t^{-\sigma}} \Pi_{j,t}^{*1-\eta} \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{1-\eta} Y_{t+\ell}
 \end{aligned}$$

Let $\tilde{\mathcal{R}}_{j,t} \equiv \mathcal{R}_{j,t} \Pi_{j,t}^{*1-\eta} C_t^{-\sigma}$, then

$$\begin{aligned}
 (B83) \quad \tilde{\mathcal{R}}_{j,t} &= \sum_{\ell=0}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{1-\eta} Y_{t+\ell} \right] \\
 (B84) \quad &= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \sum_{\ell=1}^{\infty} (\beta \theta_j)^\ell \mathbb{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{1-\eta} Y_{t+\ell} \right] \\
 (B85) \quad &= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \mathbb{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}} \right)^{1-\eta} \sum_{\ell=1}^{\infty} (\beta \theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t+1}}{P_{j,t+\ell}} \right)^{1-\eta} Y_{t+\ell} \right] \\
 (B86) \quad &= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \beta \theta_j \mathbb{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}} \right)^{1-\eta} \tilde{\mathcal{R}}_{j,t+1} \right] \\
 (B87) \quad &= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \beta \theta_j \mathbb{E}_t \left[\Pi_{j,t+1}^{\eta-1} \tilde{\mathcal{R}}_{j,t+1} \right]
 \end{aligned}$$

The discounted cost is

(B88)

$$\mathcal{C}_{j,t} = \sum_{\ell=0}^{\infty} \theta_j^\ell Q_{t,t+\ell} Y_{j,t+\ell|t} \text{MC}_{j,t+\ell|t}$$

(B89)

$$= \sum_{\ell=0}^{\infty} \theta_j^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\eta} Y_{t+\ell} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\frac{\alpha\eta}{1-\alpha}} \text{MC}_{j,t+\ell}$$

(B90)

$$= \sum_{\ell=0}^{\infty} \theta_j^\ell \left(\frac{C_{t+\ell}}{C_t} \right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \text{MC}_{j,t+\ell}$$

(B91)

$$= \frac{1}{C_t^{-\sigma}} \Pi_{j,t}^{*- \frac{\eta}{1-\alpha}} \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \text{MC}_{t+\ell}$$

Let $\tilde{\mathcal{C}}_{j,t} \equiv \mathcal{C}_{j,t} C_t^{-\sigma} \Pi_{j,t}^{* \frac{\eta}{1-\alpha}}$, then

(B92)

$$\tilde{\mathcal{C}}_{j,t} = \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \text{MC}_{t+\ell}$$

(B93)

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} Y_t \text{MC}_t + \beta\theta_j \text{E}_t \left(\frac{P_{j,t}}{P_{j,t+1}} \right)^{-\frac{\eta}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1}$$

(B94)

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} Y_t \text{MC}_t + \beta\theta_j \text{E}_t \left[\Pi_{j,t+1}^{\frac{\eta}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1} \right]$$

The steady states are

(B95)

$$\mathcal{V}_j = \mathcal{V} = \frac{Y(1 - \text{MC})}{1 - \beta\theta}$$

SOLUTIONS TO A TWO-SECTOR ECONOMY

C1. AIAO

Since the exchange rate shock is the only exogenous variable in the equation system, all endogenous variables can be expressed in terms of it:

$$(C1) \quad \tilde{y}_t = \psi_{ye}^{aiao} \Delta e_t; \quad \pi_t = \psi_{\pi e}^{aiao} \Delta e_t; \quad \hat{i}_t = \psi_{ie}^{aiao} \Delta e_t.$$

where ψ_{ye}^{aiao} , $\psi_{\pi e}^{aiao}$, and ψ_{ie}^{aiao} are unknown coefficients to be determined. Because $E_t \Delta e_{t+1} = 0$, the forecasts of the endogenous variables one period ahead are:

$$(C2) \quad E_t[\tilde{y}_{t+1}] = 0; \quad E_t[\pi_{t+1}] = 0.$$

Substitute $E_t[\tilde{y}_{t+1}]$ and $E_t[\pi_{t+1}]$ into the three-equation system, and use the Taylor rule to substitute out the nominal interest rate in the dynamic IS curve. The equation system reduces to:

$$(C3) \quad \tilde{y}_t = -\sigma^{-1} (\phi_\pi \pi_t + \phi_y \tilde{y}_t)$$

$$(C4) \quad \pi_t = \kappa \tilde{y}_t + v \Delta e_t$$

from which π_t and $\tilde{y}_t$ can be solved in terms of Δe_t :

$$(C5) \quad \tilde{y}_t = -\frac{v \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t$$

$$(C6) \quad \pi_t = \frac{v (\sigma + \phi_y)}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t$$

From the Taylor rule, the nominal interest rate is

$$(C7) \quad \hat{i}_t = \frac{v \sigma \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t.$$

Define $\Omega \equiv \sigma + \phi_y + \kappa \phi_\pi$. The coefficients are

$$(C8) \quad \psi_{ye}^{aiao} = -v \phi_\pi \Omega,$$

$$(C9) \quad \psi_{\pi e}^{aiao} = v (\sigma + \phi_y) \Omega,$$

$$(C10) \quad \psi_{ie}^{aiao} = v \sigma \phi_\pi \Omega.$$

The sectoral dynamics involve the bilateral relative price as a state variable.

$$(C11) \quad \pi_{1,t} = \psi_{\pi_{1s}}^{aiao} s_{t-1} + \psi_{\pi_{1e}}^{aiao} \Delta e_t; \quad \pi_{2,t} = \psi_{\pi_{2s}}^{aiao} s_{t-1} + \psi_{\pi_{2e}}^{aiao} \Delta e_t.$$

The forecasts for the sectoral inflation are

$$(C12) \quad E_t \pi_{1,t+1} = \psi_{\pi_1 s}^{aiao} s_t; \quad E_t \pi_{2,t+1} = \psi_{\pi_2 s}^{aiao} s_t$$

From the dollar sector's NKPC:

$$(C13) \quad \pi_{1,t} = \beta \psi_{\pi_1 s}^{aiao} s_t + \kappa \psi_{ye}^{aiao} \Delta e_t + \lambda v s_t$$

$$(C14) \quad = (\beta \psi_{\pi_1 s}^{aiao} + \lambda v) s_t + \kappa \psi_{ye}^{aiao} \Delta e_t$$

$$(C15) \quad = (\beta \psi_{\pi_1 s}^{aiao} + \lambda v) \theta s_{t-1} + [(\beta \psi_{\pi_1 s}^{aiao} + \lambda v) \theta + \kappa \psi_{ye}^{aiao}] \Delta e_t$$

The last equation comes from the autoregressive representation of the bilateral relative price. From comparison with the unknown coefficients:

$$(C16) \quad \psi_{\pi_1 s}^{aiao} = (\beta \psi_{\pi_1 s}^{aiao} + \lambda v) \theta$$

$$(C17) \quad \psi_{\pi_1 e}^{aiao} = (\beta \psi_{\pi_1 s}^{aiao} + \lambda v) \theta + \kappa \psi_{ye}^{aiao}$$

The coefficients $\psi_{\pi_1 s}^{aiao}$ and $\psi_{\pi_1 e}^{aiao}$ can be solved as:

$$(C18) \quad \psi_{\pi_1 s}^{aiao} = v (1 - \theta)$$

$$(C19) \quad \psi_{\pi_1 e}^{aiao} = v (1 - \theta) - v \kappa \phi_\pi \Omega$$

By combining the terms with common coefficients, the dollar-sector inflation can be rewritten as a function of the contemporary bilateral price and the exchange rate shock:

$$(C20) \quad \pi_{1,t} = \frac{v (1 - \theta)}{\theta} s_t - v \kappa \phi_\pi \Omega \Delta e_t$$

$$(C21) \quad = v (1 - \theta) s_{t-1} + v (1 - \theta - \kappa \phi_\pi \Omega) \Delta e_t$$

The dollar-sector inflation is above its steady state if

$$(C22) \quad 1 - \theta > \kappa \phi_\pi \Omega$$

$$(C23) \quad \sigma + \phi_y + \kappa \phi_\pi > \frac{\kappa}{1 - \theta} \phi_\pi$$

$$(C24) \quad \sigma + \phi_y > \phi_\pi (1 - \beta \theta) \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right)$$

$$(C25) \quad (1 - \beta \theta) \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) < \frac{\sigma + \phi_y}{\phi_\pi}$$

From Proposition 2, inflation in the non-dollar sector is

$$(C26) \quad \pi_{2,t} = \pi_{1,t} - \frac{1-\theta}{\theta} s_t$$

$$(C27) \quad = -\frac{(1-v)(1-\theta)}{\theta} s_t - v\kappa\phi_\pi\Omega\Delta e_t$$

The sectoral output gaps are derived from the demand functions:

$$(C28) \quad \tilde{y}_{1,t} = \tilde{y}_t + v\eta s_t$$

$$(C29) \quad = v\eta s_t - v\phi_\pi\Omega\Delta e_t$$

$$(C30) \quad = v\eta\theta s_{t-1} + v(\eta\theta - \phi_\pi\Omega)\Delta e_t$$

$$(C31) \quad \tilde{y}_{2,t} = \tilde{y}_t - (1-v)\eta s_t$$

$$(C32) \quad = -(1-v)\eta s_t - v\phi_\pi\Omega\Delta e_t$$

The contemporaneous response of the dollar-sector output gap depends on the parameters. The sector produces above the natural level if

$$(C33) \quad \eta\theta > \phi_\pi\Omega$$

$$(C34) \quad \sigma + \phi_y + \kappa\phi_\pi > \frac{\phi_\pi}{\eta\theta}$$

$$(C35) \quad \left(\frac{1}{\eta\theta} - \kappa\right)\phi_\pi < \sigma + \phi_y$$

$$(C36) \quad \frac{1}{\eta\theta} - \kappa < \frac{\sigma + \phi_y}{\phi_\pi}$$

C2. DIAO

The expected aggregate inflation in the dynamic IS curve is rewritten as:

$$(C37) \quad E_t[\pi_{t+1}] = E_t[(1-v)\pi_{1,t+1} + v\pi_{2,t+1}]$$

$$(C38) \quad = E_t[\pi_{1,t+1} + v(s_{t+1} - s_t)]$$

$$(C39) \quad = E_t[\pi_{1,t+1}] - v(1-\theta)s_t$$

where the last equation follows from (46). The bilateral relative price s_t can be viewed as an exogenous autoregressive variable in a three-equation system; so all endogenous variables can be written as functions of s_t . Let $\tilde{y}_t = \psi_{ys}^{diao} s_t$, and $\pi_{1,t} = \psi_{\pi_1 s}^{diao} s_t$. Again, it follows from (46) that $E_t[\tilde{y}_{t+1}] = \psi_{ys}^{diao} \theta s_t$, and $E_t[\pi_{1,t+1}] = \psi_{\pi_1 s}^{diao} \theta s_t$. Substitute these, together with the Taylor rule, into the dynamic IS curve and the dollar-sector NKPC yields the following equation system for ψ_{ys}^{diao}

and $\psi_{\pi_1 s}^{diao}$:

$$(C40) \quad (1 - \beta\theta) \psi_{\pi_1 s}^{diao} = \kappa \psi_{ys}^{diao} + \lambda v$$

$$(C41) \quad (\phi_\pi - \theta) \psi_{\pi_1 s}^{diao} = -[\sigma(1 - \theta) + \phi_y] \psi_{ys}^{diao} - v(1 - \theta)$$

Define $\Lambda \equiv \frac{1}{(1 - \beta\theta)[\sigma(1 - \theta) + \phi_y] + \kappa(\phi_\pi - \theta)}$. The solutions to the equation system are:

$$(C42) \quad \psi_{ys}^{diao} = -\lambda v \phi_\pi \Lambda$$

$$(C43) \quad \psi_{\pi_1 s}^{diao} = \frac{v(1 - \theta)}{\theta} (1 - \kappa \phi_\pi \Lambda)$$

The response of dollar-sector inflation depends on the parameters. It is negative if $\psi_{\pi_1 s}^{diao} < 0$, and positive otherwise. Taking into account the expression of Λ , this condition becomes:

$$(C44) \quad (1 - \beta\theta) [\sigma(1 - \theta) + \phi_y] + \kappa(\phi_\pi - \theta) < \kappa \phi_\pi$$

$$(C45) \quad (1 - \beta\theta) [\sigma(1 - \theta) + \phi_y] < \kappa \theta$$

$$(C46) \quad \lambda \sigma + \frac{(1 - \beta\theta) \phi_y}{\theta} < \kappa$$

$$(C47) \quad \phi_y < \frac{(\kappa - \lambda \sigma) \theta}{1 - \beta\theta}$$

The dynamics of the nominal interest rate are obtained by substituting the solutions to the output gap and dollar-sector inflation into the Taylor rule:

$$(C48) \quad \hat{i}_t = \phi_\pi \psi_{\pi_1 s}^{diao} s_t + \phi_y \psi_{ys}^{diao} s_t$$

$$(C49) \quad = -v(\kappa - \lambda \sigma)(1 - \theta) \phi_\pi \Lambda s_t$$

Its response is negative if:

$$(C50) \quad \kappa \sigma^{-1} > \lambda$$

$$(C51) \quad \lambda \Theta \left(1 + \frac{\sigma^{-1}(\alpha + \varphi)}{1 - \alpha} \right) > \lambda$$

$$(C52) \quad \sigma^{-1} > \frac{1 - \Theta}{\Theta} \frac{1 - \alpha}{\alpha + \varphi}$$

Inflation in the non-dollar sector is obtained using Proposition 2:

$$(C53) \quad \pi_{2,t} = \pi_{1,t} - \frac{1-\theta}{\theta} s_t$$

$$(C54) \quad = -\frac{1-\theta}{\theta} (1-v + v\kappa\phi_\pi\Lambda) s_t$$

The aggregate inflation is the weighted sum of the sectoral inflation:

$$(C55) \quad \pi_t = (1-v) \pi_{1,t} + v\pi_{2,t}$$

$$(C56) \quad = -\frac{1-\theta}{\theta} v\kappa\phi_\pi\Lambda s_t + v\Delta e_t$$

$$(C57) \quad = -(1-\theta) v\kappa\phi_\pi\Lambda s_{t-1} + v[1 - (1-\theta)\kappa\phi_\pi\Lambda] \Delta e_t$$

$$(C58)$$

The coefficient of the exchange rate shock is

$$(C59) \quad v\{\kappa\theta(\phi_\pi - 1) + (1-\beta\theta)[\sigma(1-\theta) + \phi_y]\}\Lambda > 0$$

The real interest rate is:

$$(C60) \quad \hat{r}_t = \hat{i}_t - E_t[\pi_{t+1}]$$

$$(C61) \quad = -v(\kappa - \lambda\sigma)(1-\theta)\phi_\pi\Lambda s_t + \frac{1-\theta}{\theta} v\kappa\phi_\pi\Lambda\theta s_t$$

$$(C62) \quad = v\lambda\sigma(1-\theta)\phi_\pi\Lambda s_t$$

The sectoral output gap dynamics can be derived from the demand functions:

$$(C63) \quad \tilde{y}_{1,t} = \tilde{y}_t + v\eta s_t = -v(\lambda\phi_\pi\Lambda - \eta) s_t$$

$$(C64) \quad \tilde{y}_{2,t} = \tilde{y}_t - (1-v)\eta s_t = -[\lambda v\phi_\pi\Lambda + \eta(1-v)] s_t$$

For the dollar sector to produce above the natural level, $\eta > \lambda\phi_\pi\Lambda$. Otherwise, it produces below the natural level.

C3. DIDO

Follow the same method as in the case of DIAO. Solutions to the three-equation system are:

$$(C65) \quad \tilde{y}_t = -\lambda v \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda s_t$$

$$(C66) \quad \pi_{1,t} = \frac{v(1-\theta)}{\theta} \left[1 - \kappa \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda \right] s_t$$

$$(C67) \quad \hat{i}_t = -v(\kappa - \lambda\sigma)(1-\theta) \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda s_t$$

From the IS curve, the real interest rate is positive:

$$(C68) \quad \hat{r}_t = v\lambda\sigma(1-\theta) \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda s_t$$

Inflation in the non-dollar sector is negative:

$$(C69) \quad \pi_{2,t} = -\frac{1-\theta}{\theta} \left[1 - v + v\kappa \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda \right] s_t < 0$$

Sectoral output gaps are given by:

$$(C70) \quad \tilde{y}_{1,t} = -v \left[\lambda \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda - \eta \right] s_t$$

$$(C71) \quad \tilde{y}_{2,t} = - \left[\lambda v \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda + \eta(1-v) \right] s_t$$

Aggregate inflation is:

$$(C72) \quad \pi_t = -\frac{1-\theta}{\theta} v\kappa \left(\phi_\pi + \frac{\theta \eta}{1-\theta} \phi_y \right) \Lambda s_t + v\Delta e_t$$

The coefficient of the exchange rate shock is

$$(C73) \quad v \{ \kappa\theta(\phi_\pi - 1 - \eta\phi_y) + (1 - \beta\theta)[\sigma(1-\theta) + \phi_y] \} \Lambda$$

whose sign depends on the parameters. In the case of a large elasticity of substitution, the aggregate inflation response is negative.

CURRENCY CHOICE: PRICING PROBABILITIES

In this section, we provide an analytical approximation to the probability of currency choice. For the value function of the firm, we use

$$(D1) \quad \mathcal{V}_{j,s,t} = \Xi_t(P_{j,s}^*) + E_t Q_{t,t+1} \left(\theta_j \mathcal{V}_{j,s,t+1} + (1 - \theta_j) \max_{j'} \mathcal{V}_{j',t+1,t+1} \right)$$

rather than equation (85), i.e. we exclude the continuation value of the firm, in case it can change its prices. I.e. for this appendix, we assume that firms are “reborn” with a different owner. Thus, the continuation value for the old firm owner, in the case of a price-setting opportunity, is zero. The calculations mostly carry over to the version in equation (85).

Second order approximation to the currency choice around the logged desired price is

$$(D2) \quad \Pr_{j,t} - \tilde{\Pr}_j = \frac{\gamma_j \Xi_{pp,t}}{2 \sum \gamma_{j'} \tilde{\mathcal{V}}_{j'}} E_t \left[\sum_{\ell=0}^{\infty} Q_{t,t+\ell} \left[\theta_j^\ell \left(P_{j,t}^* - \frac{\tilde{P}_{t+\ell}}{\mathcal{E}_{j,t+\ell}} \right)^2 - \sum_{j'=1}^K \tilde{\Pr}_{j'} \theta_{j'}^\ell \left(P_{j',t}^* - \frac{\tilde{P}_{t+\ell}}{\mathcal{E}_{j',t+\ell}} \right)^2 \right] \right]$$

$$(D3) \quad = \frac{\gamma_j \Xi_{pp,t}}{2 \sum \gamma_{j'} \tilde{\mathcal{V}}_{j'}} E_t \left[\sum_{\ell=0}^{\infty} \beta^\ell \left[\theta_j^\ell (p_{j,t}^* + e_{j,t+\ell} - \tilde{p}_{t+\ell})^2 - \sum_{j'=1}^K \tilde{\Pr}_{j'} \theta_{j'}^\ell (p_{j',t}^* + e_{j',t+\ell} - \tilde{p}_{t+\ell})^2 \right] \right]$$

A proposition from the baseline case of homogeneous price rigidity can be summarized as follows, echoing a result in ?:

PROPOSITION 1: *Assume homogeneous price rigidity in a two-sector economy. Let $\gamma_1 = 1$, $\gamma_2 = \gamma$. The second-order approximation to the probability of pricing in dollars is*

$$(D4) \quad \Pr_{1,t} \approx \frac{1}{1 + \gamma} + \frac{\gamma K(x_t)}{(1 + \gamma)^2 \tilde{\mathcal{V}}_t}$$

where $K(x_t) \equiv -\tilde{\Xi}_{pp}(\tilde{p}_t | x_t)$ and $\tilde{\mathcal{V}}_t$ is the value function of a firm always using the optimal flexible dollar price, both evaluated at the date- t optimal log flex price $\tilde{p}_t$.

PROOF:

The proof and its logic are patterned after ?. For a two-sector economy, let $\gamma_1 = 1$, and $\gamma_2 = \gamma$. The probability of pricing in dollars is $\Pr_t = \frac{\mathcal{V}_{1,t,t}}{\mathcal{V}_{1,t,t} + \gamma \mathcal{V}_{2,t,t}}$. A first-order approximation around the flexible-price optimum $\mathcal{V}_{1,t,t} = \mathcal{V}_{2,t,t} = \tilde{\mathcal{V}}_t$

delivers

$$\begin{aligned} \text{Pr}(\text{D5}) \quad & \widetilde{\text{Pr}} + \frac{\partial \text{Pr}_t}{\partial \mathcal{V}_{1,t,t} | \mathcal{V}_{1,t,t} = \mathcal{V}_{2,t,t} = \tilde{\mathcal{V}}_t} (\mathcal{V}_{1,t,t} - \tilde{\mathcal{V}}_t) + \frac{\partial \text{Pr}_t}{\partial \mathcal{V}_{2,t,t} | \mathcal{V}_{1,t,t} = \mathcal{V}_{2,t,t} = \tilde{\mathcal{V}}_t} (\mathcal{V}_{2,t,t} - \tilde{\mathcal{V}}_t) \\ (\text{D6}) \quad & \frac{1}{1 + \gamma} + \frac{\gamma}{(1 + \gamma)^2 \tilde{\mathcal{V}}_t} (\mathcal{V}_{1,t,t} - \mathcal{V}_{2,t,t}) \end{aligned}$$

“Telescope out,” i.e., iterate on (85) to obtain¹

$$(\text{D7}) \quad \mathcal{V}_{2,t,t} = \max_{p_{t,2}^*} E_t \left[\sum_{l=0}^{\infty} \theta^l Q_{t,t+l} \Xi(p_{s,2}^* + e_{2,t+l} | x_{t+l}) \right]$$

Consider the second-order approximation of the profit function around the log of the flex-price optimum, i.e.

$$(\text{D8}) \quad \Xi(p | x_{t+l}) \approx \tilde{\Xi}_{t+l} + \frac{1}{2} \Xi''_{t+l} (p - \tilde{p}_{t+l})^2$$

where $\tilde{\Xi}_{t+l} = \Xi(\tilde{p}_{t+l} | x_{t+l})$ is the profit at the optimal flexible price in $t + l$ and where Ξ''_{t+l} is the second derivative of $\Xi(\cdot | x_{t+l})$ at $\tilde{p}_{t+l}$. Note that the first derivative is zero by virtue of the optimality of $\tilde{p}_{t+l}$. Replace the profit function in (D7) with the second-order approximation in (D8). Additionally assume that $\Xi''_{t+l} = \Xi''_t + \epsilon_{t+l}$, where ϵ_{t+l} is (approximately) independent of other sources of randomness. Likewise, assume that $Q_{t,t+l} = \beta^l$ up to a term (approximately) independent of other sources of randomness. We obtain

$$(\text{D9}) \quad \mathcal{V}_{2,t,t} = \max_{p_{t,2}^*} E_t \left[\sum_{l=0}^{\infty} (\beta\theta)^l \left(\tilde{\Xi}_{t+l} + \frac{1}{2} \Xi''_t (p_{t,2}^* + e_{2,t+l} - \tilde{p}_{t+l})^2 \right) \right]$$

The first-order condition

$$(\text{D10}) \quad 0 = E_t \left[\sum_{l=0}^{\infty} (\beta\theta)^l \Xi''_t (p_{t,2}^* + e_{2,t+l} - \tilde{p}_{t+l}) \right]$$

can be simplified to

$$(\text{D11}) \quad p_{t,2}^* = E_t[\tilde{p}_{t+l} - e_{2,t+l}]$$

when we assume that the right-hand side is (approximately) independent of l : we

¹There is only one exchange rate, which is the exchange rate of the second currency with respect to the dollar. For notational consistency, we keep the subindex 2 on that exchange rate.

shall do so. With that (D9) becomes
(D12)

$$\mathcal{V}_{2,t,t} = E_t \left[\sum_{l=0}^{\infty} (\beta\theta)^l \tilde{\Xi}_{t+l} \right] + \Xi_t'' \sum_{l=0}^{\infty} (\beta\theta)^l \left(\frac{1}{2} \text{Var}_t[e_{2,t+l}] - \text{Cov}(e_{2,t+l}, \tilde{p}_{t+l}) + \frac{1}{2} \text{Var}_t[\tilde{p}_{t+l}] \right)$$

For the dollar, the same calculation delivers a simpler expression, since there are no exchange rate terms,

$$(D13) \quad \mathcal{V}_{1,t,t} = E_t \left[\sum_{l=0}^{\infty} (\beta\theta)^l \tilde{\Xi}_{t+l} \right] + \Xi_t'' \sum_{l=0}^{\infty} (\beta\theta)^l \frac{1}{2} \text{Var}_t[\tilde{p}_{t+l}]$$

Plug (D12) and (D13) into (D6) and rewrite to obtain equation (D4).

ADDITIONAL IMPULSE RESPONSES

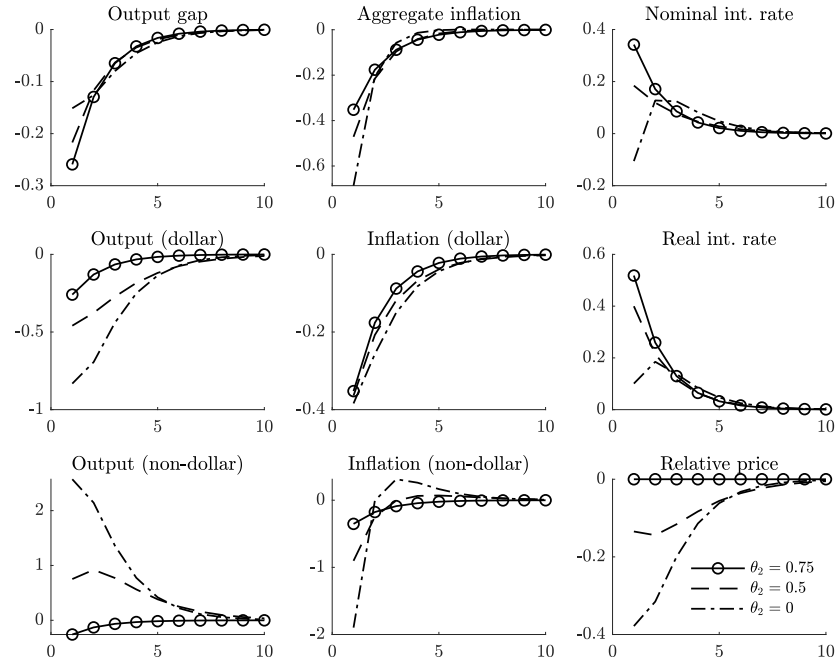


FIGURE E1. IMPULSE RESPONSES TO A 25-BASIS-POINT MONETARY POLICY SHOCK.