Supplemental Appendix:
Robustness Measures for Welfare Analysis

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A. Omitted Proofs

A1. Proof of Theorem 2

We adapt our second proof of Theorem 1 and break down the proof of Theorem 2 into the same three steps: *(i)* employing a change of variables to map the problem into an appropriate functional space; *(ii)* endowing this space with a partial order and characterizing its extremal functions; and *(iii)* mapping the solution back to the original problem. We define $\tilde{D}(\pi) = A(D(B^{-1}(\pi)))$ and $\pi = B(p)$, so that

$$\begin{aligned}\tilde{D}''(\pi) &= \frac{\mathrm{d}}{\mathrm{d}\pi}\left[\frac{A'(D(B^{-1}(\pi)))D'(B^{-1}(\pi))}{B'(B^{-1}(\pi))}\right] \\ &= \frac{1}{B'(p)}\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{A'(D(p))D'(p)}{B'(p)}\right] \in [\underline{\gamma}, \overline{\gamma}] \qquad \text{for } \pi \in [\pi_0, \pi_1].\end{aligned}$$

Throughout, we focus on the bounds implied by $\tilde{D}''(\pi) \geq \underline{\gamma}$; the bounds implied by $\tilde{D}''(\pi) \leq \overline{\gamma}$ can be similarly derived.

Step 1: Changing variables.

Instead of choosing a demand curve to maximize or minimize the loss in consumer surplus, we choose the function $h : [B(p_0), B(p_1)] \to \mathbb{R}$ defined by

$$h(\pi) := \tilde{D}'(\pi) - \underline{\gamma}\pi \qquad \text{for } \pi \in [\pi_0, \pi_1].$$

Given h, $\tilde{D}$ is completely determined, and vice versa:

$$\tilde{D}(\pi) = A(q_0) + \int_{\pi_0}^{\pi} \left[h(s) + \underline{\gamma}s\right] \,\mathrm{d}s \qquad \text{for } \pi \in [\pi_0, \pi_1].$$

This is obtained via integration by parts. Next, for some given $\underline{h}$ and $\overline{h}$, define

$$\mathscr{H}_0 = \left\{h : [\pi_0, \pi_0] \to [\underline{h}, \overline{h}] \text{ is non-decreasing}\right\}.$$

We can thus define the set of feasible functions h that are consistent with Assumption 2:

$$\mathscr{H} := \left\{ h \in \mathscr{H}_0 : h(\pi) \leq \underline{\gamma}\pi,\ \int_{\pi_0}^{\pi_1} h(s)\,\mathrm{d}s = A(q_1) - A(q_0) - \frac{1}{2}\underline{\gamma}\left(\pi_1^2 - \pi_0^2\right) \right\}.$$

Here, we have assumed that $\underline{h} \leq \min\left\{ -\underline{\gamma}\pi_0, -\underline{\gamma}\pi_1, -\left[A(q_0) - A(q_1)\right]/\left[\pi_1 - \pi_0\right] \right\}$ and $\overline{h} = -\gamma\pi_1$, with the goal of eventually taking the limit $\underline{h} \to -\infty$. Thus, we arrive at the equivalent problem:

$$\text{(A1)} \qquad \begin{cases} \overline{\Delta \mathrm{CS}} = \sup\limits_{h \in \mathscr{H}} \displaystyle\int_{p_0}^{p_1} A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(p)} \left[h(s) + \underline{\gamma} s \right]\,\mathrm{d}s \right) \mathrm{d}p, \\ \underline{\Delta \mathrm{CS}} = \inf\limits_{h \in \mathscr{H}} \displaystyle\int_{p_0}^{p_1} A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(p)} \left[h(s) + \overline{\gamma} s \right]\,\mathrm{d}s \right) \mathrm{d}p. \end{cases}$$

STEP 2: CHARACTERIZING THE SET $\mathscr{H}$.

We now endow the set $\mathscr{H}$ with a partial order. Formally, for any two functions $h_1, h_2 \in \mathscr{H}$, we write

$$h_1 \succeq h_2 \iff \int_{\pi_0}^{\pi} h_1(s)\,\mathrm{d}s \geq \int_{\pi_0}^{\pi} h_2(s)\,\mathrm{d}s \qquad \text{for } \pi \in [\pi_0, \pi_1].$$

Analogous to Lemma 1, we show:

LEMMA A.1: *Any function $h \in \mathscr{H}$ satisfies $h^* \succeq h \succeq h_*$, where:*

(i) *if* $0 \leq \underline{\gamma} \leq 2\left[A(q_0) - A(q_1)\right]/\left[A(p_1) - A(p_0)\right]^2$*, then*

$$h^*(s) := -\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0} - \frac{\gamma}{2}\left(\pi_0 + \pi_1\right),$$

$$h_*(s) := \begin{cases} \overline{h} & \text{if } s > \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}, \\ \underline{h} & \text{if } s \leq \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}; \end{cases}$$

(ii) *if* $-2\left[A(q_0) - A(q_1)\right]/\left(\pi_1 - \pi_0\right)^2 \leq \underline{\gamma} < 0$*, then*

$$h^*(s) := -\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0} - \frac{\gamma}{2}\left(\pi_0 + \pi_1\right),$$

$$h_*(s) := \begin{cases} -\underline{\gamma} s & \text{if } s > -\frac{\underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]}}{\underline{\gamma}}, \\ \underline{h} & \text{if } s \leq -\frac{\underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]}}{\underline{\gamma}}; \end{cases}$$

(iii) *if* $\underline{\gamma} < -2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2$*, then*

$$h^*(s) := \begin{cases} -\underline{\gamma}\left[\pi_1 - \sqrt{\frac{2[A(q_1)-A(q_0)]}{\underline{\gamma}}}\right] & \text{if } s > \pi_1 - \sqrt{\frac{2[A(q_1)-A(q_0)]}{\underline{\gamma}}}, \\ -\underline{\gamma} s & \text{if } s \leq \pi_1 - \sqrt{\frac{2[A(q_1)-A(q_0)]}{\underline{\gamma}}}, \end{cases}$$

$$h_*(s) := \begin{cases} -\underline{\gamma} s & \text{if } s > -\frac{\underline{h}+\sqrt{\underline{h}^2+\underline{\gamma}^2\pi_0^2+2\underline{\gamma}\left[\underline{h}\pi_0-A(q_0)+A(q_1)\right]}}{\underline{\gamma}}, \\ \underline{h} & \text{if } s \leq -\frac{\underline{h}+\sqrt{\underline{h}^2+\underline{\gamma}^2\pi_0^2+2\underline{\gamma}\left[\underline{h}\pi_0-A(q_0)+A(q_1)\right]}}{\underline{\gamma}}; \end{cases}$$

PROOF:

When the constraint $h(\pi) \leq -\underline{\gamma}\pi$ is slack, results from the information design literature (e.g., Kang and Vondrák, 2019; Kleiner, Moldovanu and Strack, 2021) imply that $h^* \succeq h \succeq h_*$ for any $h \in \mathscr{H}$, where

$$h^*(s) := -\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0} - \frac{\gamma}{2}(\pi_0 + \pi_1),$$

$$h_*(s) := \begin{cases} \overline{h} & \text{if } s > \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{1}{2}\gamma\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}, \\ \underline{h} & \text{if } s \leq \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{1}{2}\gamma\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}. \end{cases}$$

To verify that the constraint $h(\pi) \leq -\underline{\gamma}\pi$ is slack, we require:

(a) In order for $h^*(s)$ to be as stated above,

$$-\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0} - \frac{\gamma}{2}(\pi_0 + \pi_1) \leq \min\left\{-\underline{\gamma}\pi_0, -\underline{\gamma}\pi_1\right\}.$$

Equivalently,

$$\begin{cases} \underline{\gamma}(\pi_1 - \pi_0) \geq -\frac{2\left[A(q_0) - A(q_1)\right]}{\pi_1 - \pi_0}, \\ -\underline{\gamma}(\pi_1 - \pi_0) \geq -\frac{2\left[A(q_0) - A(q_1)\right]}{\pi_1 - \pi_0}. \end{cases}$$

Clearly, these inequalities hold when

$$-2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2 \leq \underline{\gamma} \leq 2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2.$$

We therefore conclude that, when these inequalities hold,

$$h^*(s) = -\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0} - \frac{\gamma}{2}(\pi_0 + \pi_1).$$

(b) In order for $h_*(s)$ to be as stated above,

$$\frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}} \in [\pi_0, \pi_1]$$

and

$$\overline{h} \le -\underline{\gamma} s \quad \text{if } s > -\frac{\underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]}}{\underline{\gamma}}.$$

Equivalently,

$$\begin{cases} \underbrace{\overline{h}}_{=-\underline{\gamma}\pi_1}(\pi_1 - \pi_0) + A(q_0) - A(q_1) + \dfrac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right) \ge 0, \\ \underbrace{\underline{h}}_{\le -\underline{\gamma}\pi_0}(\pi_1 - \pi_0) + A(q_0) - A(q_1) + \dfrac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right) \le 0, \end{cases} \quad \text{and} \quad \underline{\gamma} \ge 0.$$

These inequalities hold when $0 \le \underline{\gamma} \le 2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2$. We therefore conclude that, for $0 \le \underline{\gamma} \le \frac{2[A(q_0) - A(q_1)]}{(\pi_1 - \pi_0)^2}$,

$$h_*(s) := \begin{cases} \overline{h} & \text{if } s > \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}, \\ \underline{h} & \text{if } s \le \frac{\overline{h}\pi_1 - \underline{h}\pi_0 + A(q_0) - A(q_1) + \frac{\gamma}{2}\left(\pi_1^2 - \pi_0^2\right)}{\overline{h} - \underline{h}}. \end{cases}$$

The above argument thus proves part (i) of Lemma A.1.

Given the form of h_* stated in parts (ii) and (iii) of Lemma A.1, we next prove that $h \succeq h_*$ for any $h \in \mathscr{H}$. Let $\pi_* := -\left[\underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]}\right]/\underline{\gamma}$. Observe that $\underline{h} \le \min\left\{-\underline{\gamma}\pi_0, -\underline{\gamma}\pi_1\right\}$ implies:

$$\begin{aligned} \pi_* \ge \pi_0 &\iff \underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]} \ge -\underline{\gamma}\pi_0 \\ &\iff A(q_0) - A(q_1) \ge 0, \\ \pi_* \le \pi_1 &\iff \underline{h} + \sqrt{\underline{h}^2 + \underline{\gamma}^2\pi_0^2 + 2\underline{\gamma}\left[\underline{h}\pi_0 - A(q_0) + A(q_1)\right]} \le -\underline{\gamma}\pi_1 \\ &\iff \underline{h} \le -\frac{A(q_0) - A(q_1)}{\pi_1 - \pi_0}. \end{aligned}$$

These inequalities hold; hence $\pi_* \in [\pi_0, \pi_1]$. Then, to complete the proof of part (ii) of Lemma A.1:

- If $\pi \in [\pi_0, \pi_*]$, then the inequality $\int_{\pi_0}^{\pi} h(s)\ \mathrm{d}s \geq \int_{\pi_0}^{\pi} h_*(s)\ \mathrm{d}s$ holds trivially from the fact that $h(s) \geq \underline{h} = h_*(s)$ for $s \leq \pi_*$.
- If $\pi \in [\pi_*, \pi_1]$, then the inequality $\int_{\pi}^{\pi_1} h(s)\ \mathrm{d}s \leq \int_{\pi}^{\pi_1} h_*(s)\ \mathrm{d}s$ holds from the fact that $h(s) \leq -\underline{\gamma} s = h_*(s)$ for $s \geq \pi_*$. Since $\int_{\pi_0}^{\pi_1} h(s)\ \mathrm{d}s = \int_{\pi_0}^{\pi_1} h_*(s)\ \mathrm{d}s$, we conclude that $\int_{\pi_0}^{\pi} h(s)\ \mathrm{d}s \geq \int_{\pi_0}^{\pi} h_*(s)\ \mathrm{d}s$.

Finally, given the form of h^* as stated above in part (iii) of Lemma A.1, we assume that $\underline{\gamma} < -2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2$ and prove that $h^* \succeq h$ for any $h \in \mathscr{H}$. Now, because $\underline{\gamma} < -2\left[A(q_0) - A(q_1)\right]/(\pi_1 - \pi_0)^2$, we must have $\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}} \in [\pi_0, \pi_1]$. Then:

- If $\pi \in [\pi_0, \pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}]$, then the inequality $\int_{\pi_0}^{\pi} h^*(s)\ \mathrm{d}s \geq \int_{\pi_0}^{\pi} h(s)\ \mathrm{d}s$ holds trivially from the fact that $h^*(s) = -\underline{\gamma} s \geq h(s)$ for $s \leq \pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}$.
- If $\pi \in [\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}, \pi_1]$, then suppose there exists $\hat{\pi}$ satisfying

$$\hat{\pi} \in (\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}, \pi_1) \quad \text{and} \quad \int_{\pi_0}^{\hat{\pi}} h(s)\ \mathrm{d}s > \int_{\pi_0}^{\hat{\pi}} h^*(s)\ \mathrm{d}s.$$

Then $h(\hat{\pi}) > h^*(\hat{\pi}) = -\underline{\gamma}\left[\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}\right]$; otherwise, $h(s) \leq h^*(s)$ for every $s \in [\pi_0, \hat{\pi}]$, contradicting our assumption that $\int_{\pi_0}^{\hat{\pi}} h(s)\ \mathrm{d}s > \int_{\pi_0}^{\hat{\pi}} h^*(s)\ \mathrm{d}s$. Because h is non-decreasing, this implies that $h(s) \geq h(\hat{\pi}) > -\underline{\gamma}\left[\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}\right] = h^*(s)$ for every $s \in (\hat{\pi}, \pi_1)$. Then:

$$\begin{aligned}
\int_{\pi_0}^{\pi_1} h(s)\ \mathrm{d}s &= \int_{\pi_0}^{\hat{\pi}} h(s)\ \mathrm{d}s + \int_{\hat{\pi}}^{\pi_1} h(s)\ \mathrm{d}s \\
&> \int_{\pi_0}^{\hat{\pi}} h^*(s)\ \mathrm{d}s + \int_{\hat{\pi}}^{\pi_1} h(s)\ \mathrm{d}s \\
&\geq \int_{\pi_0}^{\hat{\pi}} h^*(s)\ \mathrm{d}s + \int_{\hat{\pi}}^{\pi_1} h^*(s)\ \mathrm{d}s = \int_{\pi_0}^{\pi_1} h^*(s)\ \mathrm{d}s.
\end{aligned}$$

This contradicts the fact that $\int_{\pi_0}^{\pi_1} h(s)\ \mathrm{d}s = \int_{\pi_0}^{\pi_1} h^*(s)\ \mathrm{d}s = A(q_1) - A(q_0) - \underline{\gamma}\left(\pi_1^2 - \pi_0^2\right)/2$ since $h, h^* \in \mathscr{H}$. Here, the first inequality follows by the definition of $\hat{\pi}$, while the second inequality follows from our observation that $h(s) > h^*(s)$ for every $s \in (\hat{\pi}, \pi_1)$. Consequently, our initial supposition was wrong: no such $\hat{\pi}$ exists; so, $\int_{\pi_0}^{\pi} h^*(s)\ \mathrm{d}s \geq \int_{\pi_0}^{\pi} h(s)\ \mathrm{d}s$ for any $\pi \in [\pi_1 - \sqrt{2\left[A(q_1) - A(q_0)\right]/\underline{\gamma}}, \pi_1]$.

This completes the proof of part (iii) of Lemma A.1.

It is easy to check that $h^*, h_* \in \mathscr{H}$. Therefore, Lemma A.1 characterizes the largest and smallest elements of the partially ordered set $(\mathscr{H}, \succeq)$.

STEP 3: MAPPING BACK TO THE ORIGINAL PROBLEM.

Having characterized the largest and smallest elements of $(\mathscr{H}, \succeq)$, it remains to map these back to the original problem. To this end, we define the functional $\Delta\mathrm{CS} : \mathscr{H} \to \mathbb{R}$ by

$$\Delta\mathrm{CS}(h) := \int_{p_0}^{p_1} A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(p)} \left[h(s) + \underline{\gamma} s\right] \,\mathrm{d}s\right) \mathrm{d}p.$$

Our problem (A1) is equivalent to maximizing and minimizing this functional over the family $\mathscr{H}$. The following lemma shows that this can be done with the aid of the partial order $\succeq$ defined in our previous step:

LEMMA A.2: *The functional $\Delta\mathrm{CS}(\cdot)$ is increasing in the partial order $\succeq$; that is, for any $h_1 \succeq h_2$,*

$$\int_{p_0}^{p_1} A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(p)} \left[h_1(s) + \underline{\gamma} s\right] \,\mathrm{d}s\right) \mathrm{d}p$$
$$\geq \int_{p_0}^{p_1} A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(p)} \left[h_2(s) + \underline{\gamma} s\right] \,\mathrm{d}s\right) \mathrm{d}p.$$

PROOF:

The result follows straightforwardly from the definition of the partial order $\succeq$, the fact that A (and hence A^{-1}) is increasing, and a pointwise comparison of the two integrands.

Together, Lemmas 1 and 2 imply that the functional $\Delta\mathrm{CS}(\cdot)$ is maximized at h^* and minimized at h_*:

$$\overline{\Delta\mathrm{CS}} = \Delta\mathrm{CS}(h^*) \quad \text{and} \quad \underline{\Delta\mathrm{CS}} = \Delta\mathrm{CS}(h_*).$$

Through straightforward computation and taking the limit $\underline{h} \to -\infty$, we obtain the result of Theorem 2.

A2. Proof of Proposition 1

Similar to the proofs of Theorems 1 and 2, we prove Proposition 1 in three steps. We focus on part (a) of Proposition 1.

STEP 1: CHANGING VARIABLES.

Let $\pi = B(p)$ be defined on $[\pi_0, \pi_1] = [B(p_0), B(p_1)]$, and consider $\tilde{D} : [\pi_0, \pi_1] \to \mathbb{R}$ be defined by $\tilde{D}(B(p)) = A(D(p))$. We choose the gradient function $\beta(\cdot)$:

$$\beta(\pi) := \tilde{D}'(\pi) \qquad \text{for } \pi \in [\pi_0, \pi_1].$$

Given $\beta(\cdot)$, $\tilde{D}(\cdot)$ is completely determined, and vice versa:

$$\tilde{D}(\pi) = A(q_0) + \int_{\pi_0}^{\pi} \beta(s)\,\mathrm{d}s \qquad \text{for } \pi \in [\pi_0, \pi_1].$$

This is obtained via integration by parts, which assumes that $\tilde{D}(\cdot)$ is absolutely continuous on $[\pi_0, \pi_1]$. Analogous to the family of demand curves $\mathscr{D}$, we define the set of feasible gradient functions:

$$\mathscr{B} := \left\{\beta : [\pi_0, \pi_1] \to [\underline{\beta}, \overline{\beta}] \text{ s.t. } \int_{\pi_0}^{\pi_1} \beta(s)\,\mathrm{d}s = A(q_1) - A(q_0)\right\}.$$

Thus we arrive at the equivalent problem:

$$\text{(A2)} \qquad \begin{cases} \overline{q} = \sup\limits_{\beta \in \mathscr{B}} A^{-1}\left(A(q_0) + \displaystyle\int_{\pi_0}^{B(\hat{p})} \beta(s)\,\mathrm{d}s\right), \\ \underline{q} = \inf\limits_{\beta \in \mathscr{B}} A^{-1}\left(A(q_0) + \displaystyle\int_{\pi_0}^{B(\hat{p})} \beta(s)\,\mathrm{d}s\right). \end{cases}$$

STEP 2: CHARACTERIZING THE SET $\mathscr{B}$.

Recall that, in Lemma 1, we showed that $\beta^* \succeq \beta \succeq \beta_*$ for any $\beta \in \mathscr{B}$, thereby characterizing the largest and smallest elements of the partially ordered set $(\mathscr{B}, \succeq)$. Here, β^* and β_* are as defined in the statement of Theorem 1.

STEP 3: MAPPING BACK TO THE ORIGINAL PROBLEM.

Having characterized the largest and smallest elements of $(\mathscr{B}, \succeq)$, it remains to map these back to the original problem. To this end, we define the functional $\hat{q} : \mathscr{B} \to \mathbb{R}$ by

$$\hat{q}(\beta) := A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(\hat{p})} \beta(s)\,\mathrm{d}s\right).$$

Our problem (A2) is equivalent to maximizing and minimizing this functional over the family $\mathscr{B}$. The following lemma shows that this can be done with the partial order defined previously:

LEMMA A.3: *The functional $\hat{q}(\cdot)$ is increasing in the partial order $\succeq$:*

$$A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(\hat{p})} \beta_1(s)\,\mathrm{d}s\right) \geq A^{-1}\left(A(q_0) + \int_{\pi_0}^{B(\hat{p})} \beta_2(s)\,\mathrm{d}s\right) \quad \textit{for any } \beta_1 \succeq \beta_2.$$

PROOF:

The result follows straightforwardly from the definition of the partial order $\succeq$, the fact that A (and hence A^{-1}) is increasing, and a pointwise comparison of the two integrands.

Together, Lemma 1 and Lemma A.3 imply that the functional $\hat{q}(\cdot)$ is maximized at β^* and minimized at β_*: $\overline{q} = \hat{q}(\beta^*)$ and $\underline{q} = \hat{q}(\beta_*)$. This completes the proof of part (a) of Proposition 1; part (b) of Proposition 1 can be proven very similarly and is therefore omitted.

B. Extensions of Robustness Measures

In this supplemental appendix, we show how our robustness measures can be extended when more complexity is allowed for. Motivated by applications in Section IV, we focus on four extensions: *(i)* counterfactual exercises; *(ii)* more observations; *(iii)* measurement error; and *(iv)* other welfare measures.

B1. Counterfactual Exercises

Our robustness measures can be extended to counterfactual exercises where only one point on the demand curve is observed. So far, we have assumed that two points on the demand curve are observed: (p_0, q_0) and (p_1, q_1). However, in counterfactual exercises such as our application in Section IV.B, the quantity that would be demanded at p_1 is not known.

To illustrate, we focus on extending our robustness measure r^* to this setting. Following our approach in Section III, it suffices to establish the analog of Theorem 1:

THEOREM B.1: *Suppose that only (p_0, q_0) and p_1 are observed. Under Assumption 1, the largest and smallest possible losses in consumer surplus between p_0 and p_1, $\overline{\Delta\mathrm{CS}}$ and $\underline{\Delta\mathrm{CS}}$, are respectively:*

$$\begin{cases} \overline{\Delta\mathrm{CS}} := \displaystyle\int_{p_0}^{p_1} A^{-1}\Big(A(q_0) + \overline{\beta}\,[B(p) - B(p_0)]\Big)\ \mathrm{d}p, \\ \underline{\Delta\mathrm{CS}} := \displaystyle\int_{p_0}^{p_1} A^{-1}\Big(A(q_0) + \underline{\beta}\,[B(p) - B(p_0)]\Big)\ \mathrm{d}p. \end{cases}$$

Theorem B.1 can be shown using our earlier geometric argument for Theorem 1, illustrated in Figure B.1. The largest possible value of $A(q_1)$ that is consistent with Assumption 1 can be found by drawing the (blue) straight line with gradient $\overline{\beta}$ that passes through the point $(B(p_0), A(q_0))$, and then finding the (red) point on the line at $B(p_1)$. It is clear that this value of q_1 must also yield the maximal $\overline{\Delta\mathrm{CS}}$; hence $\overline{\Delta\mathrm{CS}}$ must be attained by the red curve. A symmetric argument shows that $\underline{\Delta\mathrm{CS}}$ must be attained by the green curve.

B2. More Observations

Our robustness measures can also be extended to settings where more than two points on the same demand curve are observed, as is the case in some empirical applications. Doing so requires a generalization of our robustness measures to an arbitrary (finite) number of observations, which we denote by $(p_0, q_0), \ldots, (p_{n-1}, q_{n-1})$.

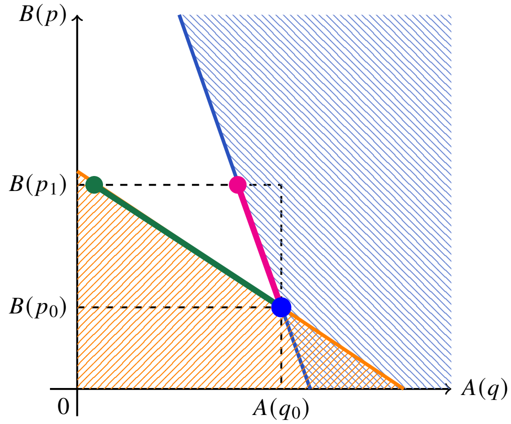

FIGURE B.1. ILLUSTRATION OF BOUNDS WHEN ONLY (p_0, q_0) AND p_1 ARE OBSERVED.

To illustrate, we again focus on extending our robustness measure r^* to this setting. The generalization of Theorem 1 to this setting is:

THEOREM B.2: *Suppose that $(p_0, q_0), \ldots, (p_{n-1}, q_{n-1})$ are observed. Define the auxiliary functions $\beta^*, \beta_* : [B(p_0), B(p_1)] \to \mathbb{R}$ as follows: for each $j \in \{0, 1, \ldots, n-1\}$,*

$$\beta^*(s) := \begin{cases} \underline{\beta} & \text{if } \frac{\overline{\beta}B(p_j) - \underline{\beta}B(p_{j+1}) - A(q_j) + A(q_{j+1})}{\overline{\beta} - \underline{\beta}} < s \leq B(p_{j+1}), \\ \overline{\beta} & \text{if } B(p_j) < s \leq \frac{\overline{\beta}B(p_j) - \underline{\beta}B(p_{j+1}) - A(q_j) + A(q_{j+1})}{\overline{\beta} - \underline{\beta}}; \end{cases}$$

$$\beta_*(s) := \begin{cases} \overline{\beta} & \text{if } \frac{\overline{\beta}B(p_{j+1}) - \underline{\beta}B(p_j) + A(q_j) - A(q_{j+1})}{\overline{\beta} - \underline{\beta}} < s \leq B(p_{j+1}), \\ \underline{\beta} & \text{if } B(p_j) < s \leq \frac{\overline{\beta}B(p_{j+1}) - \underline{\beta}B(p_j) + A(q_j) - A(q_{j+1})}{\overline{\beta} - \underline{\beta}}. \end{cases}$$

Under Assumption 1, the largest and smallest possible losses in consumer surplus between p_0 and p_1, $\overline{\Delta \mathrm{CS}}$ and $\underline{\Delta \mathrm{CS}}$, are respectively:

$$\begin{cases} \overline{\Delta \mathrm{CS}} = \int_{p_0}^{p_1} A^{-1} \left(A(q_0) + \int_{B(p_0)}^{B(p)} \beta^*(s) \, \mathrm{d}s \right) \mathrm{d}p, \\ \underline{\Delta \mathrm{CS}} = \int_{p_0}^{p_1} A^{-1} \left(A(q_0) + \int_{B(p_0)}^{B(p)} \beta_*(s) \, \mathrm{d}s \right) \mathrm{d}p. \end{cases}$$

Theorem B.2 can be shown by applying Theorem 1 between every two adjacent points. Figure B.2 illustrates the geometric argument for the case of $n = 3$ observations, where both the largest (in red) and smallest (in green) possible losses in consumer surplus are depicted.

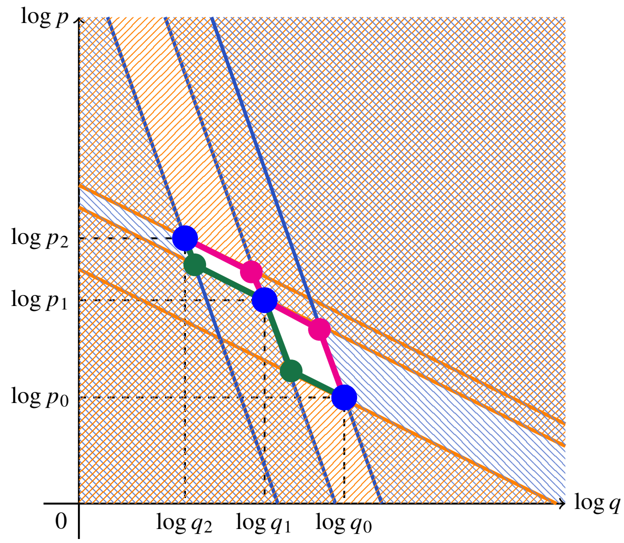

FIGURE B.2. ILLUSTRATION OF BOUNDS WITH $n = 3$ OBSERVATIONS.

B3. Measurement Error

Our robustness measures can be extended to account for uncertainty due to measurement error. Following our discussion in Section I, we focus on the uncertainty in the treatment effect estimate, $\hat{\beta}$. By propagating this uncertainty into the bounds implied by Assumptions 1 and 2—for example, by applying a bootstrap procedure to equations (4) and (5)—we can extend Theorems 1 and 2 to account for measurement error in $\hat{\beta}$. This is straightforward because equations (4) and (5) are explicit expressions of $\hat{\beta}$. It can be readily verified that these expressions are monotone with respect to $\hat{\beta}$, which implies that more precise measurements of $\hat{\beta}$ would lead to narrower bounds. In turn, we can derive corresponding confidence intervals for our robustness measures.[1]

B4. Other Welfare Measures

We have so far focused on measures of robustness for estimates of consumer surplus, defined as the integral of the Marshallian demand curve. To conclude this section, we discuss how our analysis extends to various alternative welfare measures: *(i)* deadweight loss; *(ii)* equivalent variation (EV) and compensating variation (CV); and *(iii)* supply-side welfare measures such as producer surplus.

[1]In some empirical applications, it may be possible that measurement error leads to a confidence interval for $\hat{\beta}$ that includes 0. In the spirit of our framework, nonnegative estimates of β correspond to $r^* = 1$ and $\kappa^* = \infty$: all curves that pass through the implied points $(p_0, \hat{q}_0)$ and $(p_1, \hat{q}_1)$ cannot overturn the welfare conclusion.

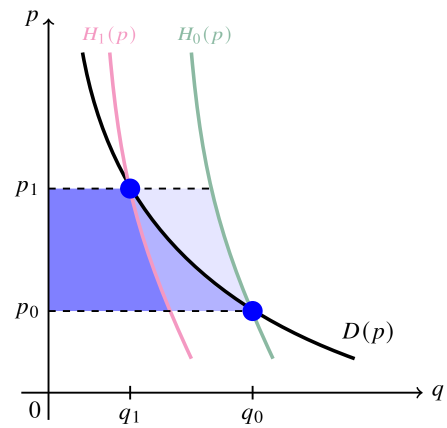

Figure B.3. Illustration of EV and CV relative to ΔCS for a normal good.

Deadweight loss.

Under additional supply-side assumptions, our results can be extended when the welfare measure of interest is deadweight loss. For example, when the supply curve is flat and producers are price takers (see Section IV.A for an empirical application), the change in deadweight loss due to a tariff $\tau = p_1 - p_0$ is

$$\Delta\mathrm{DWL} = \int_{p_0}^{p_1} D(p)\,\mathrm{d}p - (p_1 - p_0)\,q_1 = \Delta\mathrm{CS} - (p_1 - p_0)\,q_1.$$

Since (p_0, q_0) and (p_1, q_1) are observed, maximizing or minimizing ΔDWL is equivalent to maximizing or minimizing ΔCS. As such, analogs of Theorems 1 and 2 continue to hold.

EV and CV.

When consumer utility is quasilinear in money, there are no income effects and the change in consumer surplus coincides exactly with the EV and CV. However, for markets in which income effects are significant, our framework can be adapted to examine the EV and CV directly. To see this, note that the EV and CV can be defined as follows for a normal good:

$$\mathrm{EV} := \int_{p_0}^{p_1} H_1(p)\,\mathrm{d}p \quad \text{and} \quad \mathrm{CV} := \int_{p_0}^{p_1} H_0(p)\,\mathrm{d}p,$$

where H_1 and H_0 respectively denote the Hicksian demand curves at the utility levels obtained at p_1 and p_0. Figure B.3 plots an illustration of the Hicksian demand curves relative to the Marshallian demand curve considered in Sections II and III. The EV corresponds to the most darkly shaded area, left of H_1; the change in consumer surplus corresponds to the shaded area left of D as before; and the CV is the entire shaded area, left of H_0.

As noted by Willig (1976), the change in consumer surplus offers a one-sided bound to EV and CV. As Figure B.3 illustrates, when $p_1 > p_0$, EV $\geq \Delta$CS $\geq$ CV (as these welfare measures are negative). This suggests that the robustness measures for ΔCS discussed in Section III can apply as conservative measures of robustness for EV (if in the benchmark, $\Delta\text{CS} \geq G$) or CV (otherwise) as well. When this is not sufficient, Theorems 1 and 2 can be applied directly to the Hicksian demand curves H_0 and H_1 instead. Note, however, that since the counterfactual expenditures—$e(p_0, u_1)$ for EV and $e(p_1, u_0)$ for CV—are not observed, the points $(p_0, H_1(p_0))$ for CV and $(p_1, H_0(p_1))$ for EV must be treated as counterfactuals as in Supplemental Appendix B.B1.

Producer surplus.

Our results also extend straightforwardly to supply-side welfare measures like producer surplus when producers are price takers. Section IV.C considers such an empirical application where individuals supply labor. In this case, the relevant integrals are with respect to an upward-sloping supply curve, rather than a downward-sloping demand curve; but the remainder of the exercise is much the same.

C. Shape Constraints

C1. Common Shape Constraints

Different literatures in economics employ a variety of constraints on the shape of demand that capture other intuitions pertaining to their fields of interest. To be comprehensive, we consider a range of shape constraints that are considered standard in different fields. Each shape constraint (abbreviated by "SC") restricts ΔCS in a different way. We detail these assumptions below and provide some examples of how they are invoked in different fields.

(SC1) **Marshall's second law.** Demand is said to satisfy Marshall's second law if its price elasticity $\varepsilon(p) = pD'(p)/D(p)$ is decreasing in p. This was introduced by Marshall (1890) and is widely used in international trade, macroeconomics, and microeconomics, including by Krugman (1979), Bishop (1968), Johnson (2017), and Melitz (2018), who also provides some empirical justification for this shape constraint in the context of trade models.

(SC2) **Decreasing marginal revenue.** Let $P(q) \coloneqq D^{-1}(q)$ denote the inverse demand curve. Demand exhibits decreasing marginal revenue if marginal revenue $\text{MR}(q) \coloneqq P(q) + qP'(q)$ is decreasing in q. This shape constraint is standard in microeconomics (see Robinson, 1933, for example) and ensures that a profit-maximizing price exists for a monopolist who faces a convex cost function.

(**SC3**) **Log-concave demand.** Demand is log-concave if $D'(p)/D(p)$ is decreasing in p. The comprehensive surveys of Bagnoli and Bergstrom (2005) and An (1998) demonstrate that many common demand curves are log-concave. Log-concave demand also has a simple economic interpretation, as Amir, Maret and Troege (2004) show: the pass-through rate of a change in a monopolist's marginal cost is less than one if and only if demand is log-concave (see also Weyl and Fabinger, 2013). It is also well-known that log-concavity is a sufficient condition for a unique equilibrium to exist in common models of Cournot competition (Dixit, 1986) and differentiated products Bertrand competition (Caplin and Nalebuff, 1991*a*).

(**SC4**) **Concave demand.** Demand is concave if $D'(p)$ is decreasing in p. Robinson (1933) shows that concave demand has a simple economic interpretation: total output increases when third-degree price discrimination by a monopolist causes prices to rise in markets with concave demands (see also Malueg, 1994 and Inaki Aguirre, Simon Cowan and John Vickers, 2010 for variations and generalizations of this result).

(**SC5**) ρ**-concave demand.** For any given real number $\rho \in \mathbb{R}$, demand is ρ-concave if $D'(p)\,[D(p)]^{\rho-1}$ is decreasing in p. Based on the work of Prékopa (1973), this shape constraint was introduced to the economics literature by Caplin and Nalebuff (1991*b*,*a*) as a generalization of log-concavity ($\rho = 0$) and concavity ($\rho = 1$). Different values of ρ parametrize the restrictiveness of this constraint: a ρ'-concave demand curve is ρ''-concave for any $\rho'' < \rho'$.

(**SC6**) **Convex demand.** Demand is convex if $D'(p)$ is increasing in p. Similar to concave demand (**SC4**), Robinson (1933) shows that total output increases when third-degree price discrimination by a monopolist causes prices to fall in markets with convex demands (see also Malueg, 1994 and Inaki Aguirre, Simon Cowan and John Vickers, 2010 for variations and generalizations of this result).

(**SC7**) **Log-convex demand.** Demand is log-convex if $D'(p)/D(p)$ is increasing in p. Similar to log-concave demand (**SC3**), Amir, Maret and Troege (2004) show that the pass-through rate of a change in a monopolist's marginal cost is more than one if and only if demand is log-convex.

(**SC8**) ρ**-convex demand.** For any given real number $\rho \in \mathbb{R}$, demand is ρ-convex if $D'(p)\,[D(p)]^{\rho-1}$ is increasing in p. Similar to ρ-concave demand (**SC5**), ρ-convexity generalizes convexity ($\rho = 1$) and log-convexity ($\rho = 0$); a ρ'-convex demand curve is ρ''-convex for any $\rho'' > \rho'$.

These shape constraints can be divided into two categories: *concave-like* shape constraints (**SC1**)–(**SC5**) and *convex-like* shape constraints (**SC6**)–(**SC8**). Concave-like and convex-like shape constraints, respectively, bound the curvature of the demand curve from above and from below.

These shape constraints are not mutually disjoint. For example, it is well known that concave demand curves are log-concave, and that log-convex demand curves are convex. In fact:

$$\begin{tikzcd} & (\textbf{SC1}) \\ (\textbf{SC4}) \arrow[r, Rightarrow] & (\textbf{SC3}) \arrow[u, Rightarrow] \arrow[d, Rightarrow] \\ & (\textbf{SC2}) \end{tikzcd} \qquad \text{and} \qquad (\textbf{SC7}) \Longrightarrow (\textbf{SC6}).$$

These relationships are proven below in Supplemental Appendix C.C2. In Supplemental Appendix C.C3, we provide examples of common demand curves that satisfy each shape constraint.

C2. Relationships Between Assumptions

$$(\textbf{SC4}) \implies (\textbf{SC3})$$

PROOF:

Given a concave demand curve $D(\cdot)$, suppose on the contrary that there exist $p_H > p_L$ such that

$$\frac{D'(p_H)}{D(p_H)} > \frac{D'(p_L)}{D(p_L)} \implies D(p_L)D'(p_H) > D(p_H)D'(p_L).$$

Since $D(\cdot)$ is concave, $D'(p_H) \le D'(p_L)$; since $D(\cdot)$ is decreasing, $D'(\cdot) \le 0$ and $D(p_L) \ge D(p_H)$. Thus,

$$D(p_L)D'(p_H) \le D(p_H)D'(p_H) \le D(p_H)D'(p_L).$$

This is a contradiction. Hence $D(\cdot)$ is log-concave.

$$(\textbf{SC3}) \implies (\textbf{SC1})$$

PROOF:

For any $p_H > p_L$, log-concavity implies that

$$\frac{D'(p_H)}{D(p_H)} \le \frac{D'(p_L)}{D(p_L)} \implies \frac{p_H D'(p_H)}{D(p_H)} \le \frac{p_L D'(p_H)}{D(p_H)} \le \frac{p_L D'(p_L)}{D(p_L)}.$$

Here, we have used the fact that $D'(\cdot) \le 0$ as $D(\cdot)$ is decreasing. Since the above inequalities hold for any $p_H > p_L$, it follows that $D(\cdot)$ satisfies Marshall's second law.

$$(\mathbf{SC3}) \implies (\mathbf{SC2})$$

PROOF:

For any $p_H > p_L$, log-concavity implies that

$$\frac{D'(p_H)}{D(p_H)} \leq \frac{D'(p_L)}{D(p_L)} \implies p_H + \frac{D(p_H)}{D'(p_H)} \geq p_L + \frac{D(p_L)}{D'(p_L)}.$$

Since this holds for any $p_H > p_L$, it follows that $D(\cdot)$ has a decreasing marginal revenue curve.

$$(\mathbf{SC7}) \implies (\mathbf{SC6})$$

PROOF:

For any $p_H > p_L$, log-convexity implies that

$$\frac{D'(p_H)}{D(p_H)} \geq \frac{D'(p_L)}{D(p_L)} \implies D(p_L)D'(p_H) \geq D(p_H)D'(p_L).$$

Since $D(\cdot)$ is decreasing, $D'(\cdot) \leq 0$ and $D(p_L) \geq D(p_H)$. Thus

$$D(p_H)D'(p_H) \geq D(p_L)D'(p_H) \geq D(p_H)D'(p_L) \implies D'(p_H) \geq D'(p_L).$$

Since this holds for any $p_H > p_L$, it follows that $D(\cdot)$ is convex.

C3. Common Demand Curves

We now review some common demand curves that satisfy these shape constraints.

(i) Isoelastic demand curves. Each isoelastic demand curve is parametrized by its elasticity $\varepsilon \leq 0$:

$$D(p) = q_0 \left(\frac{p}{p_0}\right)^{\varepsilon}.$$

Because elasticity is constant, it must also be trivially decreasing. Hence any isoelastic demand curve satisfies Marshall's second law (**SC1**).

(ii) Constant marginal revenue demand curve. Analogous to a CES demand curve, each constant marginal revenue demand curve is parametrized by its marginal revenue $0 \leq \mu < p_0$:

$$D(p) = \frac{q_0\,(p_0 - \mu)}{p - \mu}.$$

Because marginal revenue is constant, it must also be trivially decreasing. Hence each constant marginal revenue demand curve exhibits decreasing marginal revenue (**SC2**).

(iii) *Exponential demand curves.* Each exponential demand curve is parametrized by $\lambda \geq 0$:

$$D(p) = q_0 \exp\left[-\lambda (p - p_0)\right].$$

Observe that the logarithm of any exponential demand curve is linear in p:

$$\log D(p) = \log q_0 - \lambda (p - p_0).$$

Hence each exponential demand curve is both log-concave (**SC3**) and log-convex (**SC7**).

(iv) *Linear demand curves.* Each linear demand curve is parametrized by $\lambda \geq 0$:

$$D(p) = q_0 - \lambda (p - p_0).$$

Each linear demand curve is both concave (**SC4**) and convex (**SC6**).

(v) *ρ-linear demand curves.* Each ρ-linear demand curve is parametrized by $\lambda \geq 0$:

$$D(p) = \left[q_0 - \lambda (p - p_0)\right]^{1/\rho}.$$

Each ρ-linear demand curve is both ρ-concave (**SC5**) and ρ-convex (**SC8**).

D. Empirical Application Details

We now provide additional details on our empirical applications in Section IV.

D1. Trade Tariffs

In this section, we provide the technical details behind our application to the deadweight loss of trade tariffs. To obtain the data for our exercise, we follow Amiti, Redding and Weinstein's (2019) data appendix to obtain a comprehensive dataset of products hit by new tariffs during 2018. Products are denoted by a ten-digit Harmonized Tariff Schedule (HTS10) product code and by country or origin. The dataset contains a unit quantity and total import value for each product, along with a tariff amount for each month in 2017 and 2018.

As the first step of our exercise, we replicate Amiti, Redding and Weinstein's log-log regression used to estimate the relationship between prices and quantities, assuming that prices change proportionally to tariffs within the same market (product-calendar month) between 2017 and 2018. Following Amiti, Redding and Weinstein, we estimate the regression:

$$\text{(D1)} \qquad \log\left(\frac{q_{ijt}}{q_{ij(t-12)}}\right) = \Delta \log(1 + \tau_{ijt}) + \mathrm{FE}_i + \mathrm{FE}_j + \eta_{ijt},$$

where i denotes an HTS10 product code, j denotes a country-year, t denotes a month and $\Delta \log(1 + \tau_{ijt})$ denotes the log change in the relevant tariff. This yields the elasticity estimate $\hat{\varepsilon} = -5.89$ with standard error 0.59, as reported in column (3) of Table 1 in their paper. We then follow Amiti, Redding and Weinstein in imputing the potential outcome for each q_0 based on an isoelastic curve:

$$\log(\hat{q}_{ij(t-12)}) = \log(q_{ijt}) - \hat{\varepsilon}\Delta \log((1+\tau_{ijt})). \tag{D2}$$

To compute the deadweight loss under the linear, exponential, and isoelastic curves in Figure 6, we compute the change in consumer surplus directly using the formulas in Table 1 and subtract $q_1 \times (p_1 - p_0)$ as discussed in Supplemental Appendix B.B4.[2] In each case, we treat q_1, p_1 for each good using the calendar month in 2018 as period 1 and the same month in 2017 as period 0. We then impute $p_0 = \frac{p_1}{1+\tau}$ and $\hat{q_0}$ based on equation (D2) and plug these values into the formulas directly. Following Amiti, Redding and Weinstein, we compute the deadweight loss for each market separately and then aggregate across all markets in our sample. To compute the deadweight loss under a ρ-linear demand curve, we integrate over the curve $D(p) = [q_0 - \lambda (p - p_0)]^{1/\rho}$ for each value of ρ. The formula for this is given by:

$$\mathrm{DWL}_\rho = \frac{\rho\,(p_1 - p_0)\left(q_0^{1+\rho} - q_1^{1+\rho}\right)}{(1+\rho)\left(q_0^{\rho} - q_1^{\rho}\right)} - q_1\,(p_1 - p_0)\,.$$

To compute standard confidence bands, we apply the delta method with respect to the standard error in $\hat{q_0}$ due to $\hat{\varepsilon}$ as in the motivating example. Finally, to calculate the bounds with respect to elasticity relaxations in Figure 7, we compute the lower bound on the change in consumer surplus for the isoelastic benchmark, as in the third column of Table 1 and subtract $q_1 \times (p_1 - p_0)$. We compute standard errors using the delta method with respect to the standard error of $\hat{q}_0$.

D2. Energy Subsidies

In this subsection, we provide a detailed derivation of our robustness exercise with respect to the welfare conclusion in Hahn and Metcalfe (2021). Following the description of the setting in Section IV.B,[3] the welfare effect of the CARE program is given by:

$$\Delta W = N_C \int_{q_C^*}^{q_C} [P_C(q) - \mathrm{MSC}]\ \mathrm{d}q + N_N \int_{q_N^*}^{q_N} [P_N(q) - \mathrm{MSC}]\ \mathrm{d}q - A.$$

[2]Note that Amiti, Redding and Weinstein apply an additional approximation argument before imputing a linear demand curve for each market. As they explain in footnote 9 (pp. 199–200), they make use of a second Taylor approximation in computing deadweight loss:

$$-\log(m_1/m_0) \approx (m_0 - m_1)/m_1,$$

where m_t is the total import value of a product in year t. In general, it can be shown that this approximation will underestimate deadweight loss: $-\log z \leq 1/z - 1$ for any $z \in \mathbb{R}$. As the magnitudes of the tariffs are substantial, we find that this approximation shrinks the deadweight loss estimates substantially and makes the comparison across assumptions more difficult to interpret. As such, we skip this approximation step in our calculations and instead present the deadweight loss estimates from linear (and other) interpolations using just the quantities and prices produced in their first step.

[3]See also equation (A3) in Hahn and Metcalfe's online appendix.

Here N_C and N_N are the numbers of CARE and non-CARE consumers, respectively; $P_C(\cdot)$ and $P_N(\cdot)$ are their respective inverse demand curves; and A is the administrative cost of the program. We assume that q_N and q_C are observed and that $p^* = P_N(q_N) = P_C(q_C)$. Moreover, we use Hahn and Metcalfe's equation (5) to relate p^* to the observed prices p_N and q_N and the observed quantities q_N and q_C:

$$p^* = \frac{N_N p_N q_N + N_C p_C q_C - A}{N_N q_N + N_C q_C}.$$

Summarizing:

1) We observe p^* and the points (p_N, q_N) and (p_C, q_C).
2) Hahn and Metcalfe estimate $\hat{\varepsilon}_C$ and take $\hat{\varepsilon}_N$ from Auffhammer and Rubin (2018).
3) We wish to examine how robust ΔW is to the functional form assumptions imposed by Hahn and Metcalfe and Auffhammer and Rubin. We therefore introduce two parameters, r_C and r_N, and consider

$$\begin{cases} \dfrac{\hat{\varepsilon}_C}{1-r_C} \leq \varepsilon_C \leq (1-r_C)\,\hat{\varepsilon}_C, \\ \dfrac{\hat{\varepsilon}_N}{1-r_N} \leq \varepsilon_N \leq (1-r_N)\,\hat{\varepsilon}_N. \end{cases}$$

Note that because Hahn and Metcalfe assume linear demand, it would be natural to consider relaxations of gradient variability, rather than elasticity variability. This would allow us to test the robustness of their linear benchmark directly. However, Hahn and Metcalfe provide elasticity estimates, not gradients. As such, we must decide which benchmark to use: (i) a linear benchmark using gradients inferred from elasticity estimates through the linear function or (ii) an isoelastic benchmark using the estimated elasticities directly. For our application, we choose the latter option. Our reasoning is that an isoelastic benchmark prioritizes the decisions that the authors made with respect to their exposition of price treatment effects. Hahn and Metcalfe chose to present their results in terms of an elasticity; although they could have extrapolated to a linear curve directly, interpreting their LATE estimate as a gradient, they did not do so. This decision may reflect important considerations that we would like our robustness measures to preserve.[4]

The largest possible ΔW is attained when the welfare gains from CARE households are maximized and the welfare losses from non-CARE households are minimized. Symmetrically, the smallest possible ΔW is attained when the welfare gains from CARE households are minimized and the welfare losses from non-CARE households are maximized. Importantly, we can consider these welfare effects separately since the counterfactual quantities q_C^* and q_N^* are independent of each other (as they lie on separate demand curves). Notice

[4]For instance, the LATE estimator in Hahn and Metcalfe's example uses different baseline usage numbers for CARE consumers (22.9 therms/month) than their counterfactual welfare exercise (310 therms/year), which considers a different program duration and accounts for the full CARE consumer base. Using an elasticity estimate, which is unitless, may therefore reflect an intention to accommodate the difference in magnitudes between the two quantities.

also that the additional costs, MSC and A, do not change our earlier analysis. This is because: *(i)* instead of prices p^*, p_C, and p_N, we can perform our earlier analysis on *net* prices $p^* - \text{MSC}$, $p_C - \text{MSC}$, and $p_N - \text{MSC}$; and *(ii)* the administrative cost A is simply an additive constant.

CARE Households.

Let the welfare gains for CARE households be denoted by

$$\Delta W_C = N_C \int_{q_C^*}^{q_C} [P_C(q) - \text{MSC}] \, \mathrm{d}q.$$

We want to find the largest possible values of welfare gains for CARE households, ΔW_C. We proceed in two steps: (1) we consider the problem for a given q_C^*; and (2) we optimize over the possible values of q_C^*. Throughout, we maintain the assumption that $\underline{\varepsilon} \leq \varepsilon(\cdot) \leq \overline{\varepsilon}$ on $p \in [p_C, p^*]$.

Step #1: Fixing q_C^*.

For a given q_C^*, the upper bound of ΔW_C is given by:

$$\overline{\Delta W_C}(q_C^*) = N_C \max_{P \in \mathscr{P}} \int_{q_C^*}^{q_C} [P_C(q) - \text{MSC}] \, \mathrm{d}q.$$

Our previous results (cf. Theorem 1) imply that the extremal demand curves are 2-piecewise isoelastic, with elasticities equal to $\underline{\varepsilon}$ and $\overline{\varepsilon}$ and an average elasticity of $\log(q_C/q_C^*)/\log(p_C/p^*)$. The upper bound is attained by the inverse demand curve:

$$\overline{P_C}(q; q_C^*) = \begin{cases} p^* \left(\dfrac{q}{q_C^*}\right)^{1/\underline{\varepsilon}} & \text{for } q_C^* \leq q \leq \hat{q}, \\ p_C \left(\dfrac{q}{q_C}\right)^{1/\overline{\varepsilon}} & \text{for } \hat{q} \leq q \leq q_C, \end{cases}$$

where

$$\hat{q} = \exp\left[\frac{\overline{\varepsilon}\underline{\varepsilon} \log(p_C/p^*) + \overline{\varepsilon} \log q_C^* - \underline{\varepsilon} \log q_C}{\overline{\varepsilon} - \underline{\varepsilon}}\right].$$

To obtain $\Delta W_C(q_C^*)$ at a given level of r_C, we can integrate $\overline{P_C}(q; q_C^*)$ over $q \in [q_C^*, q_C]$, substituting in $\overline{\varepsilon} = \hat{\varepsilon}_C\,(1 - r_C)$ and $\underline{\varepsilon} = \hat{\varepsilon}_C/(1 - r_C)$.

Step #2: Optimizing over q_C^*.

To optimize over q_C^*, we first determine the range of values $[\underline{q_C^*}, \overline{q_C^*}]$ that q_C^* can take, given p^*, p_C, q_C, $\hat{\varepsilon}_C$, and r_C:

$$\begin{cases} \overline{q_C^*} = q_C \left(\dfrac{p^*}{p_C}\right)^{\hat{\varepsilon}_C/(1-r_C)}, \\ \underline{q_C^*} = q_C \left(\dfrac{p^*}{p_C}\right)^{\hat{\varepsilon}_C(1-r_C)}. \end{cases}$$

The largest value of welfare losses for non-CARE households is attained by maximizing $\overline{\Delta W_C}(q_C^*)$ over $q_C^* \in [\underline{q_C^*}, \overline{q_C^*}]$. Notice, however, that the objective function is concave in q_C^*; hence $q_C^* = \underline{q_C^*}$ or $q_C^* = \overline{q_C^*}$. The largest value of welfare gains for CARE households is therefore attained by maximizing $\underline{\Delta W_C}(q_C^*)$ over $q_C^* \in [\underline{q_C^*}, \overline{q_C^*}]$. For each value of r_C, we can find the maximizing value of $\underline{\Delta W_C}(q_C^*)$ through a standard numerical optimization procedure (e.g., through a grid search).

Non-CARE Households.

Let the welfare losses for non-CARE households be denoted by

$$\Delta W_N = N_N \int_{q_N^*}^{q_N} [P_N(q) - \text{MSC}] \, \mathrm{d}q.$$

We want to find the smallest possible values of welfare losses for non-CARE households, ΔW_N. We proceed in two steps: (1) we consider the problem for a given q_N^*; and (2) we optimize over the possible values of q_N^*. Throughout, we maintain the assumption that $\underline{\varepsilon} \leq \varepsilon(\cdot) \leq \overline{\varepsilon}$ on $p \in [p_C, p^*]$.

Step #1: Fixing q_N^*.

For a given q_N^*, the lower bound of ΔW_N is given by

$$\underline{\Delta W_N}(q_N^*) = N_N \min_{P \in \mathscr{P}} \int_{q_N}^{q_N^*} [P_N(q) - \text{MSC}] \, \mathrm{d}q.$$

Again, our previous results (cf. Theorem 1) imply that the extremal demand curves are 2-piecewise isoelastic, with elasticities equal to $\underline{\varepsilon}$ and $\overline{\varepsilon}$ and an average elasticity of

$\log(q_N/q_N^*)/\log(p_N/p^*)$. The lower bound is attained by the inverse demand curve:

$$\underline{P_N}(q;q_N^*) = \begin{cases} p_N\left(\dfrac{q}{q_N}\right)^{1/\overline{\varepsilon}} & \text{for } q_N \le q \le \hat{q}, \\ p^*\left(\dfrac{q}{q_N^*}\right)^{1/\underline{\varepsilon}} & \text{for } \hat{q} \le q \le q_N^*, \end{cases}$$

where

$$\hat{q} = \exp\left[\frac{\overline{\varepsilon}\underline{\varepsilon}\log\left(p_N/p^*\right) - \underline{\varepsilon}\log q_N + \overline{\varepsilon}\log q_N^*}{\overline{\varepsilon} - \underline{\varepsilon}}\right].$$

As in the CARE case, we can obtain $\Delta W_N(q_N^*)$ by integrating $\underline{P_N}(q;q_N^*)$ over $q \in [q_N, q_N^*]$ and substituting in $\overline{\varepsilon} = \hat{\varepsilon}_N\,(1-r_N)$ and $\underline{\varepsilon} = \hat{\varepsilon}_N/(1-r_N)$.

Step #2: Optimizing over q_N^*.

To optimize over q_N^*, we first determine the range of values $[\underline{q_N^*}, \overline{q_N^*}]$ that q_N^* can take, given p^*, p_N, q_N, $\hat{\varepsilon}_N$, and r_N:

$$\begin{cases} \overline{q_N^*} = q_N\left(\dfrac{p^*}{p_N}\right)^{\hat{\varepsilon}_N/(1-r_N)}, \\ \underline{q_N^*} = q_N\left(\dfrac{p^*}{p_N}\right)^{\hat{\varepsilon}_N(1-r_N)}. \end{cases}$$

The largest value of welfare losses for non-CARE households is attained by maximizing $\overline{\Delta W_N}(q_N^*)$ over $q_N^* \in [\underline{q_N^*}, \overline{q_N^*}]$. Notice, however, that the objective function is convex in q_N^*; hence $q_N^* = \underline{q_N^*}$ or $q_N^* = \overline{q_N^*}$. The smallest value of welfare losses for non-CARE households is there attained by minimizing $\underline{\Delta W_N}(q_N^*)$ over $q_N^* \in [\underline{q_N^*}, \overline{q_N^*}]$, and we can solve for the lower bound at each r_N through a standard bounded numerical optimization procedure like grid search, as well.

Combining Bounds for Analysis.

In order to create Figure 9, we compute the upper bound on welfare gains for CARE consumers $\overline{\Delta W_C}$ and the lower bound of welfare losses for non-CARE consumers $\underline{\Delta W_N}$ for each pair of indices $(r_C, r_N) \in [0,1]\times[0,1]$. We then plot $\Delta W = N_C \cdot \overline{\Delta W_C} + N_N \cdot \underline{\Delta W_N} - A$. In each case, we use the numbers from Online Appendix B in Hahn and Metcalfe: $N_N = 3.85$M, $N_C = 1.6$M, $q_N = 490$, $q_C = 310$, $p_N = 0.95$, $p_C = 0.75$, and $p^* = 0.90$.

D3. Old-Age Pensions

In this subsection, we provide the technical details behind our application to the MVPF of the Old-Age Pensions Act based on Giesecke and Jäger (2021). The key empirical result underlying our analysis is the marginal propensity to retire early, estimated by Giesecke and Jäger through a regression discontinuity design. As a first step to our analysis, we derive a micro-foundation for interpreting this as a casual response to an increase in the pension amount through the supply curves of the eligible population.

Neoclassical Labor-Leisure Model.

We begin by considering the neoclassical labor-leisure model. Suppose that an individual i has utility over C and L, where C is consumption of goods (measured in dollars) and L is hours of leisure. We assume that utility is quasilinear with respect to consumption (as in Diamond, 1998):

$$u_i(L) + C.$$

The individual's budget constraint is

$$C \leq w_i\,(T_i - L) + V_i,$$

where T_i is total hours available, w_i is the wage rate, and V_i is other income for that individual.

To allow for the possibility of retiring in exchange for a pension, we augment this model by assuming that each individual can choose either to work ($y_i = 1$) or not ($y_i = 0$), but (for simplicity) cannot choose how much time they work. Therefore, if an individual chooses to work ($y_i = 1$), they work for $T_i - L_i$ hours. However, if an individual chooses not to work ($y_i = 0$), they receive a pension p. This changes the individual's budget constraint:

$$C \leq \begin{cases} w_i\,(T_i - L_i) + V_i & \text{if } y_i = 1, \\ p + V_i & \text{if } y_i = 0. \end{cases}$$

Summarizing, each individual faces the utility maximization problem:

$$\max\left\{u_i(L_i) + w_i\,(T_i - L_i) + V_i,\, u_i(T_i) + p + V_i\right\}.$$

Labor Supply.

Under this model, an individual chooses to retire ($y_i = 0$) if and only if

$$u_i(T_i) + p \geq u_i(L_i) + w_i\,(T_i - L_i) \iff p \geq \underbrace{u_i(L_i) - u_i(T_i) + w_i\,(T_i - L_i)}_{=:\varepsilon_i}.$$

Let the aggregate distribution of ε_i in the population be denoted by F. In theory, a fraction q_t of people retires when the pension is p_t, where

$$q_t = \mathbf{E}\left[\mathbf{1}_{p_t \geq \varepsilon_i}\right] = F(p_t).$$

We interpret F as a supply curve for retirement: ε_i is the pension that individual i would have to be paid in order for him to be indifferent to retiring.

Welfare Impact of a Pension Increase.

Suppose that the pension increases from p_0 to p_1. The change in each individual's surplus is then given by

$$\begin{aligned}\Delta W_i &= \max\left\{u_i(L_i) + w_i\left(T_i - L_i\right), u_i(T_i) + p_1\right\} \\ &\quad - \max\left\{u_i(L_i) + w_i\left(T_i - L_i\right), u_i(T_i) + p_0\right\} \\ &= \max\left\{0, p_1 - \varepsilon_i\right\} - \max\left\{0, p_0 - \varepsilon_i\right\}.\end{aligned}$$

Integrating over the population with Q_0 individuals yields

$$\begin{aligned}\Delta W &= Q_0 \int_{-\infty}^{\infty} \left[\max\left\{0, p_1 - \varepsilon_i\right\} - \max\left\{0, p_0 - \varepsilon_i\right\}\right] \, \mathrm{d}F(\varepsilon_i) \\ &= Q_0 \int_{-\infty}^{p_1} \left(p_1 - \varepsilon_i\right) \, \mathrm{d}F(\varepsilon_i) - \int_{-\infty}^{p_0} \left(p_0 - \varepsilon_i\right) \, \mathrm{d}F(\varepsilon_i) \\ &= Q_0 \int_{-\infty}^{p_1} F(\varepsilon_i) \, \mathrm{d}\varepsilon_i - \int_{-\infty}^{p_0} F(\varepsilon_i) \, \mathrm{d}\varepsilon_i \implies \Delta W = Q_0 \int_{p_0}^{p_1} F(\varepsilon_i) \, \mathrm{d}\varepsilon_i.\end{aligned}$$

Robustness With Respect to Gradients.

Our analysis builds on Giesecke and Jäger's baseline result (Section 4.1) that labor supply dropped by 6 percentage points, from 46% to 40%, at the eligibility cutoff age upon the introduction of old-age pensions in the U.K. Mapping this result to our framework, these estimates correspond to measurements of $F(\cdot)$ at two points, namely, $q_0 = \hat{F}(0s) = 0.54$ and $q_1 = \hat{F}(260s) = 0.60$. Because Q_0 is a constant (and cancels out in the MVPF calculation), we focus only on the supply curve F.

The authors include a welfare analysis in their online appendix, in which they take the extreme stance that any worker who was willing to retire at the observed pension would have been willing to retire at *any* non-zero pension. This assumption corresponds to the lower bound of any welfare measure in our framework at the limiting measure of variability (e.g., $r = 1$). As such, we conduct our robustness analysis with respect to a benchmark that is calibrated to their empirical exercise.

Because the treatment effects estimated by the regression discontinuity design are in levels-space, we focus on a linear benchmark and consider variability in gradients. Theorem 1 allows us to derive the upper and lower bounds at each r; hence we can apply the formulas from Table 1 to compute the bounds. Finally, to compute the MVPF at each r, we follow Giesecke and Jäger's Online Appendix D and divide the welfare gain at r by 1.13 to account for the net government cost of supplying each pension. To obtain confidence bands, we apply the delta method to each MVPF bound with respect to the standard error on the 6 percentage point treatment effect estimate as provided by Giesecke and Jäger.

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