Supplemental Appendix:
Optimal Security Design for Risk-Averse Investors

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Supplemental Appendix A: Reinvestment is Never Strictly Optimal

As noted in Section 3.1, one can extend the model by allowing the issuer to use capital collected from investors in order to increase some payments to other investors. Then, the feasibility constraint becomes:

$$\sum_{\theta \in \{l,h\}} f_\theta \, R_\theta(x) \leq x + \sum_\theta f_\theta t_\theta - c.$$

In order to demonstrate that such a strategy is never strictly optimal, we consider two cases.

Case 1: Suppose that only aggressive investors participate in the optimal mechanism and that the issuer raises $c + \Delta$, where $\Delta > 0$. Then, the optimal issued security must be a solution to the following problem:

$$\begin{aligned} & \min_{R_h} \left\{ f_h \int_X R_h'(x)(1 - H(x))dx \right\} \\ \text{s.t.} & \int_X R_h'(x) g_h (1 - H(x))dx = \frac{c + \Delta}{f_h} \\ & R_h'(x) \geq 0, & \forall x \in X \\ & f_h \int_0^x R_h'(z)dz \leq x + \Delta, & \forall x \in X \\ & R_h(0) = 0 \end{aligned}$$

Using the same arguments as that in the proof for Proposition 1, we can deduce that the solution to the above problem is a debt contract given by:

$$R_h(x) = \begin{cases} \frac{x+\Delta}{f_h} & \text{if } x \leq x^* \\ \frac{x^*+\Delta}{f_h} & \text{otherwise} \end{cases}$$

where x^* is the solution to $\int_0^{x^*} g_h(1 - H(z))\, dz = c$.

The above security is equivalent to a direct reimbursement of Δ coupled with the same debt contract as described in Proposition 1.

Case 2: Suppose the issuer raises $c+\Delta$ to fund a project whose return is governed by the distribution $H(x)$, and both investor types participate in the optimal mechanism. Analogous to Case 1, this can be reinterpreted as an alternative scenario where the designer intends to raise $c+\Delta$ to fund a project whose return is given by the distribution $H(x-\Delta)$. Technically, given that Lemma 3 remains valid, we obtain that $t_h = 1$ and that $t_l = \frac{c-f_h+\Delta}{f_l}$. The formulation of the designer's problem, excluding (IC-l) and (IR-h), becomes then:

$$
\begin{aligned}
&\min_{(R_\theta,t_\theta)_{\theta\in\Theta}} \left\{\int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx\right\} \\
&\quad \text{s.t.} \int_X R_l'(x) g_l (1-H(x))dx = t_l = \frac{c-f_h+\Delta}{f_l} \\
&\quad\quad \int_X \left[R_h'(x) - R_l'(x)\right] g_h (1-H(x))dx = t_h - t_l = 1 - \frac{c-f_h+\Delta}{f_l} \\
&\quad\quad R_h'(x), R_l'(x) \geq 0,\ \forall x \in X \\
&\quad\quad f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x + \Delta,\ \forall x \in X \\
&\quad\quad f_l R_l(0) + f_h R_h(0) \leq \Delta
\end{aligned}
$$

By using the same arguments as that in the proof for Theorem 2, we obtain that the solution to the above problem is given by

$$
R_l^*(x) = \begin{cases} \frac{x+\Delta}{f_l} \text{ for } x \leq x_l^* \\ \frac{x_l^*+\Delta}{f_l}, \text{ otherwise} \end{cases}
$$

$$
R_h^*(x) = \begin{cases} 0, \text{ for } x \leq x_l^* \\ \frac{x-x_l^*}{f_h}, \text{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^*-x_l^*}{f_h}, \text{ otherwise} \end{cases}
$$

In the above formulas, x_l^* and x_h^* are defined as in Theorem 2. Analogous to Case 1, the present situation can be interpreted as equivalent to directly reimbursing the low type agents with the additionally raised amount Δ, followed by issuing to both types the same securities as in Theorem 2.

By the above, we can conclude that it is without loss to solely concentrate on asset-backed securities.

Supplemental Appendix B: Proofs for Section 6: Extensions

Proofs for results in Section 6.1 (No-Purchasing Limits)

Proof of Theorem 3. We let

$$\phi(x) = f_h R_h'(x) + f_l R_l'(x)$$

be the slope of the offered aggregate securities as in the proof of Proposition 6. It follows that

$$\frac{1}{f_h}\left[\phi(x) - \frac{cf_l}{c-f_h}R_l'(x)\right] = R_h'(x) - \frac{f_l}{c-f_h}R_l'(x)$$

and the issuer's relaxed problem becomes:

$$\begin{aligned}
&\min_{R_h,R_l}\left\{c\int_X \phi(x)(1-H(x))dx\right\} \\
s.t. &\int_X R_l'(x)g_l(1-H(x))dx = 1 \\
&\int_X \phi(x)g_h(1-H(x))dx = \frac{f_l c}{c-f_h}\int_X R_l'(x)g_h(1-H(x))dx \\
&\phi(x) \geq f_l R_l'(x) \geq 0, && \forall x \in X \\
&c\int_0^x \phi(z)dz \leq x, && \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We solve the above problem analogously to the method we used in Section 4.2. First, fixing the security R_l, the optimal average slope ϕ is given by $\phi(x) = \mathbf{1}_{x\leq\hat{x}_h}$ where $\hat{x}_h$ is the solution to

$$\int_0^{\hat{x}_h} g_h(1-H(x))dx = \frac{f_l c}{c-f_h}\int_X R_l'(x)g_h(1-H(x))dx.$$

This yields a debt contract. Next, the seller must choose the optimal security R_l in order to minimize

$$\int_X \phi(x)(1-H(x))dx$$

This is equivalent to

$$\begin{aligned}
&\min_{R_l}\left\{\int_X R_l'(x)g_h(1-H(x))dx\right\} \\
s.t. &\int_X R_l'(x)g_l(1-H(x))dx = \frac{c-f_h}{f_l} \\
&\int_0^x R_l'(z)dz \leq \frac{1}{f_l}\min\{x,\hat{x}_h\}, \forall x \in X
\end{aligned}$$

Since g_l is a convex transformation of g_h, then, again by the same argument as in Section 4.2, we obtain that the optimal security R_l satisfies $R_l'(x) = \frac{1}{f_l}\mathbf{1}_{x \leq \hat{x}_l}$ where $\hat{x}_l$ is the solution to the equation

$$\int_0^{\hat{x}_l} g_l(1 - H(x))dx = c - f_h.$$

It can be then easily computed that $\hat{x}_h$ solves

$$\int_0^{\hat{x}_h} g_h(1 - H(x))dx = \frac{c}{c - f_h} \int_0^{\hat{x}_l} g_h(1 - H(x))dx.$$

which is equivalent to

$$\frac{1}{f_h} \int_{\hat{x}_l}^{\hat{x}_h} g_h(1 - H(x))dx = \frac{1}{c - f_h} \int_0^{\hat{x}_l} g_h(1 - H(x))dx$$

as desired. It follows that the optimal securities R_l and R_h are now given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l}, \text{ for } x \leq \hat{x}_l \\ \frac{\hat{x}_l}{c - f_h}, \text{ otherwise} \end{cases}$$

and

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \leq \hat{x}_l \\ \frac{x - \hat{x}_l}{f_h}, \text{ for } \hat{x}_l \leq x \leq \hat{x}_h \\ \frac{\hat{x}_h - \hat{x}_l}{f_h}, \text{ otherwise} \end{cases}$$

The omitted (IC-l) constraint holds by the same argument as in the proof for Theorem 2. ■

Lemma 5 *The interest rate for senior debt* $\frac{\hat{x}_l}{c - f_h} - 1$ *is greater than the interest rate for junior debt* $\frac{\hat{x}_h - \hat{x}_l}{f_h} - 1$*, in the case without purchase limits.*

Proof. To prove that this is indeed the case, recall that the (IC-h) constraint now reads:

$$\int_X [R_h'(x) - R_l'(x)]\, g_h(1 - H(x))dx = 0$$
$$\Leftrightarrow \frac{1}{f_h} \int_{\hat{x}_l}^{\hat{x}_h} g_h(1 - H(x))dx = \frac{1}{c - f_h} \int_0^{\hat{x}_l} g_h(1 - H(x))dx$$

As the function $g_h(1 - H(x))$ is non-negative and decreasing, the following chain of inequalities immediately follows from (IC-h):

$$\frac{g_h(1-H(\hat{x}_l))}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx > \frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx$$
$$= \frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx > \frac{g_h(1-H(\hat{x}_l))}{c-f_h}\int_0^{\hat{x}_l} dx$$

The above inequalities further imply that

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx > \frac{1}{c-f_h}\int_0^{\hat{x}_l} dx \Rightarrow \frac{\hat{x}_h-\hat{x}_l}{f_h}-1 > \frac{\hat{x}_l}{c-f_h}-1$$

as desired. ■

Proofs for results in Section 6.2 (Moral Hazard)

Proof of Proposition 4. By the Lagrangian principle there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the optimal $\bar{R}$ maximizes

$$\int_X \bar{R}'(x)\big((1+\lambda)[1-H_1(x)]-[1-H_0(x)]\big)dx = \int_X \bar{R}'(x)\big((1+\lambda)Z(x)-1\big)[1-H_0(x)]dx\,,$$

where $Z(x) = (1-H_1(x))/(1-H_0(x))$. As the distributions H_0, H_1 are ordered in the hazard rate order, Z is increasing. Furthermore, $Z(0) = 1$ such that if $\lambda \geq 0$ we get that $\bar{R}(x) = x$ which contradicts $\pi < \int xh_1(x)dx$. If $\lambda \leq -1$ this implies that $\bar{R}(x) = 0$ which contradicts $\pi > 0$. Thus, $\lambda \in (-1, 0)$ which implies that $(1+\lambda)Z(x)-1$ changes sign at most once from negative to positive which in turn implies that $\bar{R}'$ is 0 below some level and 1 above that level which means $\bar{R}$ is a call option. As H_1 admits a density, there is a unique call option with expectation π which is thus the unique solution to (1). ■

Proof of Theorem 4. Part (ii): Suppose that $\kappa_1 < \kappa^\dagger$. In this case given the securities (R_l^0, R_h^0) it is not optimal for the seller to deviate to the action $a = 0$. Thus, when taking the action $a = 1$ the seller obtains the same profit with and without moral hazard which implies that (R_l^1, R_h^1) must be optimal securities. We are left to check that the seller can not benefit from offering the securities which are optimal under the action $a = 0$ and taking the action $a = 0$. Doing so would decrease her profit as

$$\int_X (\bar{R}^0)'(1-H_0(x)) - (\bar{R}^*)_1'(1-H_1(x))dx + \kappa_1$$
$$\leq -\left[\int_X (\bar{R}^1)'(H_0(x)-H_1(x))dx - \kappa_1\right] = -[\kappa^\dagger - \kappa_1] < 0\,.$$

Part (i): If κ_1 exceeds the above bound, then since κ^* is strictly increasing, incentivizing the action $a = 1$ would require leaving the designer with a strictly higher profit than what could be achieved with the ability to commit to an action. As this is not feasible, it becomes impossible to incentivize $a = 1$. It remains to demonstrate that $a = 0$ can be incentivized. We note that the benefit of taking action $a = 0$ over action $a = 1$ is given as

$$\begin{aligned}
&\int_X (\bar{R}^0)'(1 - H_0(x))dx - \int_X (\bar{R}^0)'(1 - H_0(x))dx + \kappa_1 \\
&\qquad = -\left[\int_X (\bar{R}^0)'(H_0(x) - H_1(x))dx - \kappa_1\right] \\
&\qquad \geq -\left[\int_X (\bar{R}^1)'(H_0(x) - H_1(x))dx - \kappa_1\right] \\
&\qquad = \kappa_1 - \kappa^\dagger > 0\,.
\end{aligned}$$

Thus, the agent does not want to deviate to taking the high action given the securities (R_l^0, R_h^0) are sold to investors. ■

Proofs for Section 6.3 Security Design by a Risk-Averse Issuer

Proof of Proposition 5. Restricting attention to doubly monotonic contracts, and following essentially the same steps as above, the issuer's problem becomes

$$\begin{aligned}
&\min_{R_l, R_h} \left\{\int_X [f_l R_l'(x) + f_h R_h'(x)] g(1 - H(x))dx\right\} \\
&\text{s.t.} \int_X R_l'(x) g_l(1 - H(x))dx = \frac{c - f_h}{f_l} \\
&\qquad \int_X [R_h'(x) - R_l'(x)] g_h(1 - H(x))dx = t_h - t_l = \frac{1 - c}{f_l} \\
&\qquad R_h'(x), R_l'(x) \geq 0 \ \forall x \in X \\
&\qquad f_l R_l'(x) + f_h R_h'(x) \leq 1 \ \forall x \in X \\
&\qquad R_h(0), R_l(0) = 0
\end{aligned}$$

The third and fourth constraints represent the double monotonicity conditions. Together, the two conditions imply that the contract is feasible, and thus the feasibility constraint $f_l R_l(x) + f_h R_h(x) \leq x$ for all x is no longer needed.

By assumption, there exists an increasing and convex function $k(\cdot)$ such that $g(z) = k(g_h(z))$. It must be the case that $k(0) = 0$ and $k(1) = 1$.

As in the benchmark model, we first derive the optimal mechanism for the relaxed problem where we impose neither (IC-l) nor (IR-h). We later check that the obtained solution for the relaxed problem indeed satisfies these omitted constraints. Formally, the relaxed problem is:

$$\begin{aligned}
&\min_{R_l,R_h}\left\{\int_X [f_l R_l'(x)+f_h R_h'(x)]g(1-H(x))dx\right\}\\
&\text{s.t.} \int_X R_l'(x)g_l(1-H(x))dx=\frac{c-f_h}{f_l}\\
&\qquad \int_X [R_h'(x)-R_l'(x)]g_h(1-H(x))dx=t_h-t_l=\frac{1-c}{f_l}\\
&\qquad R_h'(x),R_l'(x)\geq 0\ \forall x\in X\\
&\qquad f_l R_l'(x)+f_h R_h'(x)\leq 1\ \forall x\in X\\
&\qquad R_h(0),R_l(0)=0
\end{aligned}$$

The proof follows a similar procedure to that of Proposition 6. We first fix R_l, and look at the following relaxed problem:

$$\begin{aligned}
&\min_{R_h}\left\{\int_X R_h'(x)g(1-H(x))dx\right\}\\
&s.t. \int_X R_h'(x)g_h(1-H(x))dx=\int_X R_l'(x)g_h(1-H(x))dx+\frac{1-c}{f_l}\\
&0\leq R_h'(x)\leq\frac{1}{f_h},\ \forall x\in X\\
&R_h(0)=0
\end{aligned}$$

Consider a new, artificial asset whose return is governed by the distribution $\tilde{H}: X\to[0,1]$ defined by

$$1-\tilde{H}(x)=g_h(1-H(x)) \text{ for all } x\in X$$

Then, the above problem can be rewritten as follows:

$$\begin{aligned}
&\min_{R_h}\left\{\int_X R_h'(x)k(1-\tilde{H}(x))dx\right\}\\
&s.t. \int_X R_h'(x)(1-\tilde{H}(x))dx=\int_X R_l'(x)(1-\tilde{H}(x))dx+\frac{1-c}{f_l}\\
&0\leq R_h'(x)\leq\frac{1}{f_h},\ \forall x\in X\\
&R_h(0)=0
\end{aligned}$$

Let

$$\tilde{V}(R_h)=\int_X R_h'(x)k(1-\tilde{H}(x))dx;\ \tilde{C}(R_h)=\int_X R_h'(x)(1-\tilde{H}(x))dx$$

denote the utility derived from holding security R_h by an agent whose dual risk preference is described by the distortion k, and the cost to a risk-neutral seller of issuing such a security, respectively.

The issuer's problem is thus equivalent to the design of a doubly monotonic security that **minimizes** the agent's utility while keeping the expected cost fixed. Then, by Theorem 1, the optimal security has the form of an equity:

$$\tilde{R}_h(x) = \begin{cases} 0 \text{ for } x \leq \tilde{x}_h \\ \frac{x-\tilde{x}_h}{f_h}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_h$ is the solution to

$$\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l}$$

By following essentially the same procedure as in the proof of Proposition 6, we obtain that the optimal R_l takes the form of senior debt, and is given by:

$$\tilde{R}_l(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \leq \tilde{x}_l \\ \frac{\tilde{x}_l}{f_l}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_l$ solves

$$\int_0^{\tilde{x}_l} g_l(1-H(x))dx = c - f_h$$

It follows that

$$\begin{aligned}\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx &= \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l} \\ &= \frac{1}{f_l}\int_0^{\tilde{x}_l} g_h(1-H(x))dx + \frac{1-c}{f_l}\end{aligned}$$

The last step is to check the menu described in Proposition 5 satisfies the ignored constraints (IR-h) and (IC-l). Note that

$$\tilde{R}_h'(x) - \tilde{R}_l'(x) = \begin{cases} -\frac{1}{f_l},\ x \leq \tilde{x}_l \\ 0,\ \tilde{x}_l \leq x \leq \tilde{x}_h \\ \frac{1}{f_h}\ x \geq \tilde{x}_h \end{cases}$$

increases on $[0, \bar{x}]$. Then, we can use the same arguments as that in the proof of Theorem 2 to show that the two ignored constraints are satisfied. ■

Case 2: The issuer solves then

$$\begin{aligned}
&\max_{R_l, R_h} f_l t_l + f_h \\
\text{s.t.} \quad &\int_X R_l'(x) g_l(1 - H(x))\, dx = t_l, \\
&\int_X [R_h'(x) - R_l'(x)] g_h(1 - H(x))\, dx = t_h - t_l = 1 - t_l, \\
&R_h'(x), R_l'(x) \geq 0 \quad \forall x \in X, \\
&f_l R_l'(x) + f_h R_h'(x) = 1 \quad \forall x \in X, \\
&R_h(0), R_l(0) = 0.
\end{aligned}$$

The constraints are respectively (IR-l), (IC-h), double monotonicity, and feasibility constraints. Note that (IC-h) can be rewritten as

$$\begin{aligned}
f_h t_l &= f_h - f_h \int_X [R_h'(x) - R_l'(x)] g_h(1 - H(x))\, dx \\
&= \int_X [1 - f_h R_h'(x) + f_h R_l'(x)] g_h(1 - H(x))\, dx = \int_X R_l'(x) g_h(1 - H(x))\, dx.
\end{aligned}$$

so the issuer's problem can be rewritten as

$$\begin{aligned}
&\min_{R_l} \int_X R_l'(x) g_h(1 - H(x))\, dx \\
\text{s.t.} \quad &0 \leq R_l'(x) \leq 1 \quad \forall x \in X, \\
&R_h(0) = 0.
\end{aligned}$$

The problem is essentially the same as the baseline model case where only one type is needed to finance the project (Section 5.1, Proposition 1). The solution is to give a debt contract to the conservative type and the remaining asset (which takes the form of equity) to the aggressive type. The issuer, who is the most risk-averse, sells all of the assets to investors and only keeps cash.

Proofs for Section 6.4 Private Budgets

Proof of Theorem 5. The proof consists of three main steps.

Step 1: Suppose that the agents' budget types are public information while the risk types remain the agents' private information, as before. We show that there exists an optimal menu such that $R^*_{l,\beta}(x) = \beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x) = \beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta} = \beta t^*_{l,1}$, and $t^*_{h,\beta} = \beta t^*_{h,1}$.

Step 1-a: We first show that if there exists an optimal mechanism $(R^*_{\theta b}, t^*_{\theta b})$ for which $R^*_{l,\beta}(x) \neq \beta R^*_{l,1}(x)$, then we can construct another optimal mechanism $(\tilde{R}^*_{\theta b}, \tilde{t}^*_{\theta b})$ such that $\tilde{R}^*_{l,\beta}(x) = \beta \tilde{R}^*_{l,1}(x)$.

Observe that in any optimal mechanism, the constraints (IR-$l1$), (IR-$l\beta$), (IC-$h1$), and (IC-$h\beta$) must all bind:

$$\begin{aligned}(IR-l1):&\int_X R'_{l,1}(x)g_l(1-H(x))dx = t_{l,1}\\(IR-l\beta):&\int_X R'_{l,\beta}(x)g_l(1-H(x))dx = t_{l,\beta}\\(IC-h1):&\int_X [R'_{h,1}(x)-R'_{l,1}(x)]g_h(1-H(x))dx = t_{h,1}-t_{l,1}\\(IC-h\beta):&\int_X [R'_{h,\beta}(x)-R'_{l,\beta}(x)]g_h(1-H(x))dx = t_{h,\beta}-t_{l,\beta}\end{aligned}$$

Putting the above equations together yields:

$$\begin{aligned}&p\int_X [R'_{h,\beta}(x)-R'_{l,\beta}(x)]g_h(1-H(x))dx+(1-p)\int_X [R'_{h,1}(x)-R'_{l,1}(x]g_h(1-H(x))dx\\&=pt_{h,\beta}+(1-p)t_{h,1}-\int_X [pR'_{l,\beta}(x)+(1-p)R'_{l,1}(x)]g_l(1-H(x))dx\\\Rightarrow&\int_X [pR'_{h,\beta}(x)+(1-p)R'_{h,1}(x)]g_h(1-H(x))dx-pt_{h,\beta}-(1-p)t_{h,1}\\&=\int_X [pR'_{l,1}(x)+(1-p)R'_{l,\beta}(x)][g_h(1-H(x))-g_l(1-H(x)]dx\end{aligned}$$

Thus, as long as the total asset assigned to conservative investors remains unchanged, i.e. as long as

$$pR^*_{l,\beta}(x)+(1-p)R^*_{l,1}(x)=p\tilde{R}^*_{l,\beta}(x)+(1-p)\tilde{R}^*_{l,1}(x)\ \forall x,$$

we can construct another incentive compatible mechanism where the total asset assigned to the aggressive investors and their total expected payment are also unchanged. If the original mechanism was optimal, so is the new one.

Step 1-b: By (IR-$l1$) and (IR-$l\beta$), in the newly constructed mechanism $(\tilde{R}^*_{\theta b},\tilde{t}^*_{\theta b})$, $\tilde{R}^*_{l,\beta}(x)=\beta\tilde{R}^*_{l,1}(x)$ implies $\tilde{t}^*_{l\beta}=\beta\tilde{t}^*_{l1}$.

Step 1-c: Suppose now that $\tilde{t}^*_{h,\beta}\neq\beta\tilde{t}^*_{h,1}$. This means that the budget of aggressive investors are not exhausted. By using similar arguments to those in Lemma 3, it can be verified that such a mechanism cannot be optimal.

Finally, steps (1.a)-(1.c) together imply that $\tilde{R}^*_{h,\beta}(x)=\beta\tilde{R}^*_{h,1}(x)$.

Step 2: By Step 1, assuming that the agents' budget types are public information, we can restrict attention to the class of menus that satisfy $R^*_{l,\beta}(x)=\beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x)=\beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta}=\beta t^*_{l,1}$, and $t^*_{h,\beta}=\beta t^*_{h,1}$. Then by following essentially the same arguments as in the proof of Theorem 2, we can show that the mechanism described in Theorem 5 is optimal in this class.

Step 3: The remaining step is to verify that, even when budget types are private information, the mechanism described in Theorem 5 is implementable, and thus optimal. It is clear that the individual rationality constraints for all types remain the same, so that they are satisfied. Moreover, as in the public budget setting, no agent has incentive to pretend to be another agent with the same budget type but different risk type. We show below that either an agent has no incentive to pretend to be another agent with the same risk type but different budget type, or he is unable to do so:

a Type $l1$ has no incentive to pretend to be of type $l\beta$ since in either case he will earn a payoff of 0 (this follows from the homogeneity of dual utility).

b Type $l\beta$ may not have enough money ($\beta < t_{l1}$) to pretend to be type $l1$. Even if $\beta > t_{l1}$, type $l\beta$ still has no incentive to pretend to be of type $l1$ since in either case he will earn a payoff of 0.

c Type $h\beta$ cannot pretend to be type $h1$ since he does not have enough money to do so ($\beta < 1 = t_{h1}$).

d Finally, type $h1$ has no incentive to pretend to be of type $h\beta$ since:

$$\begin{aligned}\int_X R'_{h,1}(x)g_h(1-H(x))dx - t_{h,1} =& \frac{1}{\beta}[\int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}] \\ >& \int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}\end{aligned}$$

Finally, no type of investor wants here to misreport in both dimensions: since an agent who misreports his budget essentially "adopts" the utility function of that budget type, the observation follows from the standard incentive compatibility constraint with respect to deviations in the risk type only. To conclude, even if budget types are private information, the mechanism described in Theorem 5 is implementable, and yields the same expected profit as in the case with public budget. Therefore, it must be an optimal mechanism. ■

Proofs for the results of Section 6.5 (More than Two Types): Here each investor is characterized by a type $\theta_n \in \Theta = \theta_1, \theta_2 ... \theta_N$, that determines his risk preferences according to distortion function g_n. Each type occurs with a probability $f_n > 0$, such that $\sum_{n=1}^{N} f_n = 1$. We further assume that the investors' risk attitudes are ordered: for each $n > 1$, g_{n-1} is a convex transformation of g_n. We consider the case where all N types are needed to finance the project, i.e., $1 - f_1 < c$, and we examine direct mechanisms $(R_n, t_n)_{n=1}^N$. Following similar arguments as those for the benchmark model, we consider mechanisms that satisfy the following constraints:

$$\sum_{n=1}^{N} f_n t_n = c. \tag{FC}$$

For a type θ_n agent not to deviate and claim to be of type θ_k, it must hold that

$$\int_X [R_n'(x) - R_k'(x)]\, g_n(1 - H(x))dx \geq t_n - t_k \tag{IC-nk}$$

Similarly, in order to ensure that a type θ_k agent purchases the security offered to him instead of pursuing an outside option that is normalized here to yield zero utility, it must be the case that

$$\int_X R_n'(x) g_n(1 - H(x))dx \geq t_n \tag{IR-n}$$

The feasibility constraint requires for each $x \in X$

$$\sum_{n=1}^{N} f_n R_n(x) \leq x$$

which is equivalent to $R_n(0) = 0$ for $\theta_n \in \Theta$ and

$$\sum_{n=1}^{N} \int_0^x f_n R_n'(z)dz \leq x \tag{Feasibility}$$

for all $x > 0$. Additionally, we require that the return of the security is increasing in the return of the underlying asset: for any $\theta_n \in \Theta$

$$R_n'(x) \geq 0 \tag{M}$$

and for all x, and that each type θ_n has a limited budget of 1:

$$t_n \leq 1. \tag{BC}$$

The designer's problem is to

$$\min_{R_1,\ldots,R_N} \sum_{n=1}^{N} \int_X f_n R_n'(x)(1 - H(x))\, dx$$

subject to all the above-mentioned constraints. By using similar arguments as those for Lemma 3, one can verify that in the optimal mechanism, $t_n^* = 1$ for all $n \geq 2$ and

$t_1^* = \frac{c-1+f_1}{f_1}$.

We first solve the following relaxed problem and then show that the solution to the relaxed problem is also the solution to the original problem.

$$
\begin{aligned}
\text{(Problem } P'\text{)} \quad & \min_{R_1,\dots,R_N} \left\{ \sum_{n=1}^{N} \int_X f_n R_n'(x)(1-H(x))\,dx \right\} \\
\text{s.t.} \quad & \int_X R_1'(x) g_1 (1-H(x))\,dx = t_1^* \\
& \int_X [R_{n+1}'(x) - R_n'(x)] g_{n+1}(1-H(x))\,dx = t_{n+1}^* - t_n^*, \quad \forall n \\
& R_n'(x) \geq 0, \quad \forall n\ \forall x \in X \\
& \sum_{n=1}^{N} f_n \int_0^x R_n'(z)\,dz \leq x, \quad \forall x \in X \\
& R_n(0) = 0, \quad \forall n; \quad t_1^* = \frac{c-1+f_1}{f_1}, \quad t_n^* = 1, \quad \forall n \geq 2
\end{aligned}
$$

Note that in the relaxed problem, we only consider (IR-θ_1) and local incentive compatibility constraints (IC-$n+1, n$) for all n (i.e., no type has incentive to pretend to be the type just below his type). We will later verify that the solution to the relaxed problem satisfies all ignored constraints, and therefore is also the solution to the original problem.

Proposition 7 *Suppose that $c > 1 - f_1$ and let x_1^* denote the solutions to:*

$$\int_0^{x_1^*} g_1(1-H(x))dx = c - 1 + f_1,$$

x_2^* denote the solutions to:

$$\frac{1}{f_2}\int_{x_1^*}^{x_2^*} g_2(1-H(x))\,dx = \frac{1}{f_1}\int_0^{x_1^*} g_2(1-H(x))\,dx + \frac{1-c}{f_1}$$

and x_n^* for any $n \geq 3$

$$\frac{1}{f_n}\int_{x_{n-1}^*}^{x_n^*} g_n(1-H(x))dx = \frac{1}{f_{n-1}}\int_{x_{n-2}^*}^{x_{n-1}^*} g_n(1-H(x))dx$$

respectively. The solution to (Problem P') is given by

$$R_1^*(x) = \begin{cases} \frac{x}{f_1}, & \text{for } x \le x_1^* \\ \frac{x_1^*}{f_1}, & \text{otherwise} \end{cases}$$

$$R_n^*(x) = \begin{cases} 0, & \text{for } x \le x_{n-1}^* \\ \frac{x - x_{n-1}^*}{f_n}, & \text{for } x_{n-1}^* \le x \le x_n^* \\ \frac{x_n^* - x_{n-1}^*}{f_n}, & \text{otherwise} \end{cases}$$

for any $n \ge 2$.

Proof for Proposition ??. The proof follows a procedure very similar to that of Proposition 6, so we omit some of the details here.

For notational convenience, let

$$\phi_n(x) = \sum_{k=1}^{n} f_k R_k'(x)$$

denote the *slope* of the total securities offered to the k lowest types. Then observe that

$$f_n[R_n'(x) - R_{n-1}'(x)] = \phi_n(x) - \sum_{k=1}^{n-1} f_k R_k'(x) - f_n R_{n-1}'(x).$$

Step 1 First, we keep $R_1, R_2, \ldots, R_{N-1}$ fixed. Then, we need to solve:

$$\min_{\phi_N} \left\{ \int_X \phi_N(x)(1 - H(x))\, dx \right\}$$

subject to:

$$\int_X \phi_N(x) g_N (1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N(1 - H(x))\, dx,$$

$$\int_0^x \phi_N(z)\, dz \le x, \quad \forall x \in X,$$

$$\phi_N(x) \ge 0, \quad \forall x \in X.$$

By the same argument as in the proof for Proposition 6 Step 1, one can show that the optimal ϕ_N is given by $\phi_N^*(x) = \mathbf{1}_{x \le x_N^*}$, where x_N^* solves

$$\int_0^{x_N^*} g_N(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N(1 - H(x))\, dx.$$

Step 2: Observe that minimizing x_N^* is equivalent to the minimization problem:

$$\min_{R_1 \ldots R_{N-1}} \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N (1 - H(x))\, dx$$

under the same constraints. We now fix $R_1, R_2, \ldots, R_{N-2}$ and solve this new minimization problem, which is further equivalent to minimizing

$$\frac{f_{N-1}}{f_N + f_{N-1}} \phi_{N-1}(x).$$

This is because all terms in $\phi_{N-1}(x)$ except R_{N-1}' are fixed, so they can be added or subtracted from the objective function. Then we need to solve:

$$\min_{\phi_{N-1}} \left\{ \int_X \phi_{N-1}(x) g_N (1 - H(x))\, dx \right\}$$

subject to:

$$\int_X \phi_{N-1}(x) g_{N-1}(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-2} f_k R_k'(x) + f_{N-1} R_{N-2}'(x) \right] g_{N-1}(1 - H(x))\, dx + t_{N-1}^* - t_{N-2}^*,$$

$$\int_0^x \phi_{N-1}(z)\, dz \leq x, \quad \forall x \in X,$$

$$\phi_{N-1}(x) \geq 0, \quad \forall x in X.$$

By the same arguments as in Step 1, we can show that the optimal ϕ_{N-1} is given by $\phi_{N-1}^*(x) = \mathbf{1}_{x \leq x_{N-1}^*}$, where x_{N-1}^* solves

$$\int_0^{x_{N-1}^*} g_{N-1}(1 - H(x))\, dx = \int_X \left[\sum_{n=1}^{N-2} f_n R_n'(x) + f_{N-1} R_{N-2}'(x) \right] g_{N-1}(1 - H(x))\, dx + t_{N-1}^* - t_{N-2}^*.$$

Recall that $t_n^* = 1$ for $n \geq 2$ and $t_1^* = \frac{c-1+f_l}{f_l}$.

Step 3-Step $N-1$ *(only needed for* $N \geq 4$*)*: Repeat the above step $N-3$ times. We can show that for any $n \geq 2$, the optimal $\phi_n^*(x)$ is given by $\mathbf{1}_{x \leq x_n^*}$, where x_n^* solves

$$\int_0^{x_n^*} g_n (1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{n-1} f_k R_k'(x) + f_n R_{n-1}'(x) \right] g_n (1 - H(x))\, dx + t_n^* - t_{n-1}^*.$$

Step N: Now our problem is reduced to choosing the optimal R_1 to

$$\min_{R_1}\left\{\int_X R_1'(x)g_2(1-H(x))\,dx\right\}$$

subject to:

$$\begin{aligned}
&\int_X R_1'(x)g_1(1-H(x))\,dx = \frac{c-1+f_1}{f_1},\\
&f_1\int_0^x R_1'(z)\,dz \le x,\\
&R_1'(x)\ge 0, \quad \forall x\in X,\\
&R_1(0)=0.
\end{aligned}$$

We can then apply the same argument as in the proof for Proposition 6, Step 2, and obtain that the solution to the above problem is $(R_1^*)'(x)=\frac{1}{f_1}\mathbf{1}_{x\le x_1^*}$, where x_1^* solves

$$\frac{1}{f_1}\int_0^{x_1^*} g_1(1-H(x))\,dx = \frac{c-1+f_1}{f_1}.$$

In order to verify that the securities described in Proposition 6 are the solution to (Problem P'), we still need to confirm that x_n^* is increasing in n to ensure that the feasibility constraint is satisfied. This follows from, for any $n\ge 2$,

$$\begin{aligned}
\int_0^{x_n^*} g_n(1-H(x))\,dx &= \int_X\left[\sum_{k=1}^{n-1} f_kR_k'(x)+f_nR_{n-1}'(x)\right]g_n(1-H(x))\,dx + t_n^*-t_{n-1}^*\\
&=\int_0^{x_{n-1}^*} g_n(1-H(x))\,dx + \int_X f_nR_{n-1}'(x)g_n(1-H(x))\,dx + t_n^*-t_{n-1}^*\\
&\ge \int_0^{x_{n-1}^*} g_n(1-H(x))\,dx.
\end{aligned}$$

The inequality follows as $t_n^*\ge t_{n-1}^*$ and $R_{n-1}'(x)\ge 0$ for all $x\in X$. This completes the proof. ■

Proof for Theorem 6. In order to prove that the mechanism described in Theorem 6 are the optimal securities, we still need to show that the omitted constraints are also satisfied.

Step 1: We show that if (IC-$n+1$, n) holds for any n, then (IC-n, k) holds for any $n>k$.

First, observe that:

$$\begin{aligned}\int_X \left[R_n'(x) - R_k'(x)\right] g_n(1-H(x))\,dx = &\int_X \left[R_n'(x) - R_{n-1}'(x)\right] g_n(1-H(x))\,dx\\ &+ \int_X \left[R_{n-1}'(x) - R_{n-2}'(x)\right] g_n(1-H(x))\,dx\\ &+ \ldots\\ &+ \int_X \left[R_{k+1}'(x) - R_k'(x)\right] g_n(1-H(x))\,dx.\end{aligned}$$

Next, take any $i > j$. Since $R_i' - R_j'$ is increasing on $[x_{j-1}^*, x_i^*]$ and g_j is more convex than g_i, we can use the same arguments as in the proof of Theorem 2 to show that:

$$\int_X \left[R_i'(x) - R_j'(x)\right] g_i(1-H(x))\,dx \geq \int_X \left[R_i'(x) - R_j'(x)\right] g_j(1-H(x))\,dx.$$

Combining these results, we obtain:

$$\begin{aligned}\int_X \left[R_n'(x) - R_k'(x)\right] g_n(1-H(x))\,dx \geq &\int_X \left[R_n'(x) - R_{n-1}'(x)\right] g_n(1-H(x))\,dx\\ &+ \int_X \left[R_{n-1}'(x) - R_{n-2}'(x)\right] g_{n-1}(1-H(x))\,dx\\ &+ \ldots\\ &+ \int_X \left[R_{k+1}'(x) - R_k'(x)\right] g_{k+1}(1-H(x))\,dx\\ \geq &\,(t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \ldots + (t_{k+1} - t_k) = t_n - t_k.\end{aligned}$$

Thus, (IC-n, k) holds as desired.

Step 2: Take any $n > k$. We show that if (IC-n, k) holds , then (IC-k, n) also holds. The proof is essentially the same as that for Theorem 2, so we omit the details.

Step 3: The fact that (IR-k) holds follows directly from (IR-1) and (IC-n, k).

Steps 1-3 together show that all omitted constraints are satisfied. Therefore, we can conclude that the solution to (Problem P') is also the solution to the original problem. That is, the securities described in Theorem 6 are optimal. ■

Supplemental Appendix C: Risk Preferences in Financial Markets

In the main paper we assumed that investors have preferences that correspond to a class of non-expected utility models. There is ample laboratory evidence showing that expected utility does not perform well in explaining agents' risk taking behavior (Bruhin, Fehr-Duda and Epper [?], Diecidue, Wakker and Zeelenberg [?] among others). Some of these papers argue that probability distortions play an important role (see the survey of Starmer [?]), and find more support for models of rank dependent preferences than for EU preferences (Weber and Kirsner [?]). Other papers question the validity of rank-dependent utility. For example, Bernheim and Sprengler [?] argue that subjects do not adjust the weights they assign to different outcomes when the ranking of the outcomes changes, as predicted by rank-dependent theory. These authors suggest that the original Prospect Theory (PT) may fit the data in experiments better than both rank-dependent and EU preferences. Yet, as it is well known, PT violates FOSD even in very simple lotteries (see Wakker [?]). Diecidue, Wakker and Zeelenberg [?] found evidence for the presence of rank dependent preferences and elicited individual weight functions. However, they also found that the probability weights change even if the ranks do not, violating rank-dependent theory. Wu [?] illustrated a violation of the ordinal independence axiom that is a necessary property of any rank-dependent preferences[30]. Oprea [?] argues that probability weighting in lotteries stems from the complexity of lottery evaluation rather than from the attitude towards risk.

Field evidence from financial markets also provides mixed support for expected utility. Barberis, Huang and Thaler [?] showed that a combination of first-order risk aversion and narrow framing can explain the stock market participation puzzle. Probability weighting can explain several financial phenomena, such as low average returns on IPO securities (Barberis and Huang [?]). Polkovnichenko [?] showed that rank-dependent preferences are consistent with observed patterns of investment in both well-diversified and poorly-diversified portfolios of stocks. Such patterns are inconsistent with any theory, such as expected utility, in which risk attitudes stem from the curvature of the utility function only. Polkovnichenko and Zhao [?] estimated probability weighting functions from option prices assuming rank-dependent utility and cumulative prospect theory preferences. While they show that agents apply probability weighting (and hence do not follow EU), they find evidence for the inverse-S shape weighting postulated in Prospect Theory.

[30] Ordinal independence states that if two lotteries share the same upper tale, then the preference between these two lotteries remains the same even if the upper tail is substituted with another common tail.

Several studies emphasized the presence of heterogeneity of risk attitudes among decision makers. Considering both expected utility and prospect theory, von Gaudecker, van Soest, and Wengström [?] found that risk preferences are heterogeneous and that most of this heterogeneity cannot be explained by observables such as age, gender and education. The importance of heterogeneity in risk preferences and probability weighting was illustrated in Andrikogiannopoulou and Papakonstantinou [?] in the context of sport betting.

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