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Supplemental Appendix (or Appendix S) for 'Asymmetric Models of Sales"

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APPENDIX S. FURTHER PROOFS AND EXTENDED RESULTS

In this appendix, we supplement and extend various results or points in the main text. We:

- S.1 report the basic properties of equilibria;
- S.2 fully construct equilibria and
- S.3 provide a summary of that process;
- S.4 and S.5 illustrate pathological equilibria in knife-edge cases;
- S.6 offer extended results with innovation choices;
- S.7 document the properties of an equilibrium with a clearinghouse;
- S.8 consider the simultaneous choices of prices and technologies; and
- S.9 provide a mapping between all-pay contests and models of sales.

S.1. **Equilibrium Properties of Simultaneous Pricing.** We now document several (relatively standard) properties that must hold for any (Nash) equilibrium. We write $F_i(p)$ for the mixed strategy of firm i. As usual, by an atom we mean a price at which $F_i(p)$ discontinuously increases. We write $\underline{p}_i$ for the infimum of the support of prices played by firm i in equilibrium.

Lemma S1 (Equilibrium Properties). *An equilibrium of a model of sales has these properties.*

(i) There are no atoms strictly below v and at most $n-1$ atoms at v.

(ii) The upper bound of the support of prices for all firms is v.

(iii) There is no gap in the joint support of firms' strategies. Relatedly, if any interval of prices is in the support for some firm i then it is in the support for some other firm $j \neq i$.

(iv) At least $n-1$ of the firms earn their captive-only profit.

(v) Profits satisfy: (a) firm n earns at least $(\lambda_n+\lambda_S)(p^\dagger_{n-1}-c_n) \geq \lambda_n(v-c_n)$ with strict inequality if $p^\dagger_n < p^\dagger_{n-1}$; (b) if $p^\dagger_n < p^\dagger_{n-1}$, then firm $i<n$ earns its captive-only profit and places an atom at v; and (c) if $p^\dagger_{n-1} < p^\dagger_{n-2}$, then firm n earns exactly $(\lambda_n+\lambda_S)(p^\dagger_{n-1}-c_n)$.

Proof of Lemma S1. Claims (i)–(iii) are relatively standard for games of this kind, and so the proofs are omitted. For completeness, they are retained in Appendix C of our working paper.

Claim (iv). All firms earn weakly more than their captive-only profit. Suppose that two or more firms earn strictly more. If all place an atom at v, the necessarily all have strictly positive probability of winning the shoppers at this price. There would be a strict incentive for at least one firm to undercut. We conclude that at least one firm j that earns more strictly more than its captive-only profit does play an atom at v, but is willing price arbitrarily close v.

Consider another firm $i \neq j$. If that firm plays an atom at v, then that atom will lose to the firm j so will sell only to captives, and so will earn its captive-only profit. If instead firm i mixes up to (but do not place an atom at) v then it is willing to price arbitrarily close to v and doing so almost always loses all shopper sales. Once again it must earn its captive-only profit.

Claim (v). Part (a) follows from an argument in the main text. Turning to part (b), by claim (iv) $n-1$ firms earn captive-only profits. Given that n earns strictly more, this must apply to all $i \in \{1, \ldots, n-1\}$. For n to earn strictly more than captive-only profits and yet be willing to price arbitrarily close to v, it must win the shoppers with probability bounded away from zero, which implies that each $i < n$ places an atom at v. Finally, for part (c), if firm n were to earn strictly more than $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$, then $\underline{p}_n > p^\dagger_{n-1}$. Firm $n-1$ could then set a price $p_{n-1} \in (p^\dagger_{n-1}, p^\dagger_{n-2})$ which would capture shoppers with certainty and earn strictly more than its captive-only profit; a contradiction of claim (iv). $\square$

We note here the properties of profits in claim (v). Firm n earning $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$ is the profit level implied by the characterization of equilibrium strategies we give below, for any model parameters. That characterization also gives a unique equilibrium profile for generic parameter values. This leaves open special cases with $p^\dagger_n = p^\dagger_{n-1}$ or $p^\dagger_{n-1} = p^\dagger_{n-2}$ or both. A continuum of other equilibria can exist, in which exactly one of $\{n, n-1, n-2\}$ does strictly better than the profits given by Proposition 1. These "pathological" payoffs are mentioned in the main text, and then set aside. In this appendix we also describe such (pathological) equilibria.

S.2. **Equilibrium Construction.** A construction here delivers equilibria with non-pathological profits (as in Proposition 1) for any parameters. We use the following notation and terminology.

Definition (Required and Minimum Win Probabilities). *The* required win probability $w_i(p)$ *is the probability with which firm i must win the business of shoppers for it to earn its equilibrium profit from the price $p \in (0, v)$. Relatedly, the* minimum win probability $\underline{w}_i(p)$ *is the probability that gives the firm its captive-only profit $\lambda_i(v - c_i)$ from charging the price p.*

In this construction we look only for non-pathological payoffs. Each firm $i < n$ earns its captive-only profit and so, $w_i(p) = \underline{w}_i(p)$. For the remaining firm, $w_n(p) \geq \underline{w}_n(p)$. We will express equilibrium mixtures in terms of required win probabilities: if p is in the support of firm i then it must sell to the shoppers with probability $w_i(p)$. If p is not in the support, then it captures that business with probability weakly less than $w_i(p)$. Recall that $\underline{w}_i(p)$ satisfies

$$\lambda_i(v - c_i) = (p - c_i)\left(\lambda_i + \lambda_S \underline{w}_i(p)\right) \quad \Rightarrow \quad \underline{w}_i(p) = \frac{\lambda_i(v - p)}{\lambda_S(p - c_i)}. \tag{S1}$$

This is decreasing in p and satisfies $\underline{w}_i(v) = 0$.

We pause briefly to relate minimum win probabilities to the ordering of firms by aggressiveness. A firm's lowest undominated price $p_i^\dagger$ satisfies $\underline{w}_i(p_i^\dagger) = 1$, which solves to give eq. (1) from the main text. We defined firm i to be (strictly) more aggressive than firm j if $p_i^\dagger < p_j^\dagger$, so firm i is willing to choose a lower price to capture shoppers. Equivalently, this holds if

$$\frac{v - c_i}{v - c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}. \tag{S2}$$

By construction $\underline{w}_j(p) > 1 \geq \underline{w}_i(p)$ for $p \in [p_i^\dagger, p_j^\dagger)$. However, a stronger aggression ranking entails $\underline{w}_j(p) > \underline{w}_i(p)$ for all $p \in [p_j^\dagger, v)$. This (partial ordering of firms) requires

$$\frac{\lambda_i}{p - c_i} < \frac{\lambda_j}{p - c_j}, \tag{S3}$$

which holds for all relevant p if and only if

$$\frac{v - c_i}{v - c_j} > \max\left\{\frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \frac{\lambda_i}{\lambda_j}\right\}. \tag{S4}$$

(We can use this to derive a weaker condition than that stated in Proposition 2 to establish the uniqueness of a "two to tango" equilibrium.) If this does not hold so that

$$\frac{\lambda_i}{\lambda_j} \geq \frac{v - c_i}{v - c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \tag{S5}$$

then i is more aggressive ($p_i^\dagger < p_j^\dagger$) and so is more willing to charge a lower price, but for sufficiently high prices firm j has a lower minimum win probability, which means that j is relatively more enthusiastic about offering a higher price. In this situation there is a unique price $p_{ij}^\dagger$ at which $\underline{w}_i(p_{ij}^\dagger) = \underline{w}_j(p_{ij}^\dagger)$, and as p rises through this point $\underline{w}_j(p)$ crosses $\underline{w}_i(p)$ from above to below. We will use this property (which corresponds to an effective change in "aggression" ranking as price increases) when we fully characterize an equilibrium below.

We now consider the required win probability for firm n. It can earn more than its captive-only profit. We have already discussed conditions under which it earns $\pi_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n) = \lambda_n(v - c_n) + \Delta_n$ where $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Its required win probability is

$$w_n(p) = \frac{\lambda_n(v - p) + \Delta_n}{\lambda_S(p - c_n)} = \underline{w}_n(p) + \frac{\Delta_n}{\lambda_S(p - c_n)}. \tag{S6}$$

We now characterize the distributions used in firms' mixed strategies. In $F_i(p)$ for firm i is continuously increasing and satisfies $F_i(p) < 1$ for $p < v$ (from (i) and (ii) of Lemma S1). We write $I(p) \subseteq \{1, \ldots, n\}$ for the firms that are on the "dance floor" at price $p \in (p_{n-1}^\dagger, v)$. These are firms where $F_i(p)$ is strictly increasing at that price.[39] If a firm is on the dance floor then at that price its expected profit must equal its equilibrium profit, or equivalently its probability of winning the shoppers must equal its required win probability $w_i(p)$. It wins the shoppers only if all other firms $j \neq i$ price above it, which happens with probability $\prod_{j \neq i}(1 - F_j(p))$. That is,

$$w_i(p) = \prod\nolimits_{j \neq i}(1 - F_j(p)) \quad \Leftrightarrow \quad 1 - F_i(p) = \frac{1 - F_S(p)}{w_i(p)}$$
$$\text{where} \quad 1 - F_S(p) \equiv \prod\nolimits_{j=1}^{n}(1 - F_j(p)), \tag{S7}$$

where we obtained the second equality after multiplying and dividing by $1 - F_i(p)$, which we are able to do given that $F_i(p) < 1$ and so $1 - F_i(p) > 0$ for $p < v$. Here $F_S(p)$ is the distribution of

[39] By which we mean that $F_i(p_L) < F_i(p_H)$ for all p_H and p_L satisfying $p_H > p > p_L$.

the cheapest price and so the distribution of prices paid by the shoppers. Relatedly, if any new firm were to charge a price p then it would win the sales of shoppers with probability $1 - F_S(p)$.

We can substitute the expression for $F_i(p)$ back into the expression for $F_S(p)$, and obtain

$$
\begin{aligned}
1 - F_S(p) = \prod_{j=1}^{n}(1 - F_j(p)) &= \prod_{i\in I(p)} \frac{1 - F_S(p)}{w_i(p)} \prod_{j\notin I(p)}(1 - F_j(p)) \\
\Leftrightarrow \quad 1 - F_S(p) &= \left(\frac{\prod_{j\in I(p)} w_j(p)}{\prod_{j\notin I(p)}(1 - F_j(p))} \right)^{1/(|I(p)|-1)} \\
\Rightarrow \quad 1 - F_i(p) &= \frac{1}{w_i(p)} \left(\frac{\prod_{j\in I(p)} w_j(p)}{\prod_{j\notin I(p)}(1 - F_j(p))} \right)^{1/(|I(p)|-1)}, \qquad \text{(S8)}
\end{aligned}
$$

where $|I(p)|$ is the number of firms actively mixing ("dancing") at p. The term $\prod_{j\notin I(p)}(1 - F_j(p))$ corresponds to firms who do not dance, and so is (locally) constant. For those firms $i \in I(p)$ who dance, we need the solutions $F_i(p)$ to be valid distribution functions that are (given that firms actively mix) strictly increasing. The density $f_i(p)$ is

$$
f_i(p) = (1 - F_i(p)) \left(\frac{w_i'(p)}{w_i(p)} - \frac{1}{|I(p)| - 1} \sum_{j\in I(p)} \frac{w_j'(p)}{w_j(p)} \right), \qquad \text{(S9)}
$$

and this is strictly positive for all $i \in I(p)$ (as required) if and only if

$$
(|I(p)| - 1) \max_{i\in I(p)} \left\{ \frac{-w_i'(p)}{w_i(p)} \right\} < \sum_{j\in I(p)} \frac{-w_j'(p)}{w_j(p)}. \qquad \text{(S10)}
$$

If $|I(p)| = 2$ (so that there is a "tango" between two firms) then this is always satisfied. However, it can fail (and, as we show, it will fail for asymmetric firms) if $|I(p)| > 2$.

We now construct an equilibrium. Such an equilibrium partitions the full "dance floor" $[p_{n-1}^{\dagger}, v)$ into at most $n-1$ subintervals. In each subinterval one firm $i \in \{1, \ldots, n-1\}$ continuously mixes (or "dances") together with firm n, and then at the top of the subinterval firm i shifts all remaining mass to v and is replaced by a substitute firm j mixing in the next subinterval. Firm n mixes over the entire interval $[p_{n-1}^{\dagger}, v)$ but (in essence) swaps dance partners at various points so that the "two to tango" property (Baye, Kovenock and De Vries, 1992) holds within each subinterval, but more than two firms can participate in randomized sales overall.

Suppose that the two most aggressive firms are distinct: $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. We have found exact equilibrium profits in this case (claim (v) of Lemma S1) and firms n and $n-1$ must mix down to $p_{n-1}^\dagger$ (if $p_n^\dagger = p_{n-1}^\dagger$ or $p_{n-1}^\dagger = p_{n-2}^\dagger$ we can also proceed with the profits implied by claim (v) of Lemma S1). We set $I(p) = \{n-1, n\}$ and so $|I(p)| = 2$ for all $p \in [p_{n-1}^\dagger, p_{n-2}^\dagger)$, and use the solutions for the mixing distributions reported in eq. (S8), which simplify to $F_n(p) = 1 - w_{n-1}(p)$ and $F_{n-1}(p) = 1 - w_n(p)$ and where $F_S(p) = 1 - w_{n-1}(p)w_n(p)$. We note that

$$1 - F_S(p) = w_{n-1}(p)w_n(p) < w_{n-1}(p) < 1 \leq w_i(p) \tag{S11}$$

for all $p \in (p_{n-1}^\dagger, p_{n-2}^\dagger]$ and all $i \in \{1, \ldots, n-2\}$. This means that no other firm wishes to "join the dance floor" at a price $p_{n-2}^\dagger$ and just above. Thus we continue to apply the solutions here as p increases through $p_{n-2}^\dagger$. One possibility is that $1 - F_S(p) < w_i(p)$ for all $p \in [p_{n-2}^\dagger, v)$ and all $i \in \{1, \ldots, n-2\}$. If so, then we have constructed a unique equilibrium in which firms $n-1$ and n "tango" over $[p_{n-1}^\dagger, v)$ while all other firms strictly prefer to maintain $p_i = v$. We note that the solutions reported here satisfy $\lim_{p \uparrow v} F_n(p) = 1$.

The other possibility is that we reach a price at which $1 - F_S(p) = w_i(p)$ for some firm $i \in \{1, \ldots, n-2\}$ so that firm i wishes to "step on to the dance floor" to join the tango. Without loss of generality, we label the firms so that it is firm $n-2$ that wishes to join the dance floor and we write $p_{n-2}^\ddagger$ for the (lowest) price at which this happens. For generic parameter choices, firm $n-2$ is uniquely defined and so our construction will be unique. If there is more than one firm that wishes to "join in" then we pick a firm for which $w_j(p)$ is falling most rapidly, so that $w'_{n-2}(p_{n-2}^\ddagger) \leq w'_j(p_{n-2}^\ddagger)$ for any other firm $j \in \{1, \ldots, n-3\}$ where $w_j(p_{n-2}^\ddagger) = w_{n-2}(p_{n-2}^\ddagger)$. There can be (non-generic) circumstances in which we have multiple choices available. One such situation is when two firms i and j are pairwise symmetric in the sense that $\lambda_i = \lambda_j$ and $c_i = c_j$, and in this circumstance our choice of firm that "steps in" is arbitrary; there are multiple equilibria in this case. For our chosen (generically unique) firm $n-2$,

$$w_{n-2}(p_{n-2}^\ddagger) = 1 - F_S(p_{n-2}^\ddagger) < w_{n-1}(p_{n-2}^\ddagger). \tag{S12}$$

This means that $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below within the interval $(p_{n-1}^\dagger, p_{n-2}^\ddagger)$. Given that the minimum win probability functions can cross only once (as established earlier) we can conclude that $w_{n-1}(p) > w_{n-2}(p)$ for all $p \in (p^{\ddagger}_{n-2}, v)$. This means (as we will confirm) that once firm $n-2$ joins the dance floor, firm $n-1$ will strictly prefer to stay off it.

We continue the construction for prices above $p^{\ddagger}_{n-2}$. We set $F_{n-1}(p) = F_{n-1}(p^{\ddagger}_{n-2})$ for all $p \in [p^{\ddagger}_{n-2}, v)$ so that firm $n-1$ leaves the dance floor and places remaining mass at v. We then set $I(p) = \{n-2, n\}$ (and so we maintain $|I(p)| = 2$) for prices at and (at least locally) above $p^{\ddagger}_{n-2}$. These firms then mix according to eq. (S8) where these solutions satisfy

$$F_n(p) = 1 - \frac{w_{n-2}(p)}{w_n(p^{\ddagger}_{n-2})} \quad \text{and} \quad F_{n-2}(p) = 1 - \frac{w_n(p)}{w_n(p^{\ddagger}_{n-2})} \quad \Rightarrow$$
$$1 - F_S(p) = (1 - F_n(p))(1 - F_{n-1}(p^{\ddagger}_{n-2}))(1 - F_{n-2}(p)) = \frac{w_n(p) w_{n-2}(p)}{w_n(p^{\ddagger}_{n-2})}. \tag{S13}$$

We apply these solutions for prices rising above $p^{\ddagger}_{n-2}$ until a price (discussed below) at which we see another "partner swapping event." Before we do this, however, we perform two checks.

Firstly, we consider whether firm $n-2$ could join the dance floor to form a threesome rather than replacing firm $n-1$, so that $I(p) = \{n-2, n-1, n\}$ at $p^{\ddagger}_{n-2}$ and just above. Given that $|I(p)| = 3$, then inequality of eq. (S10) required for positive densities is

$$2 \min_{i \in \{n-2, n-1, n\}} \left\{ \frac{w_i'(p)}{w_i(p)} \right\} > \sum_{j \in \{n-2, n-1, n\}} \frac{w_j'(p)}{w_j(p)}. \tag{S14}$$

A necessary condition for this to hold is

$$\frac{w_{n-2}'(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} \geq \sum_{j \in \{n-1, n\}} \frac{w_j'(p^{\ddagger}_{n-2})}{w_j(p^{\ddagger}_{n-2})}. \tag{S15}$$

However, we know that $w_{n-2} > 1 - F_S(p) = w_n(p) w_{n-1}(p)$ for $p < p^{\ddagger}_{n-2}$ but with equality at $p = p^{\ddagger}_{n-2}$, and so $w_{n-2}(p) - w_n(p) w_{n-1}(p)$ is decreasing at $p^{\ddagger}_{n-2}$. That is,

$$w_{n-2}'(p^{\ddagger}_{n-2}) < w_n(p^{\ddagger}_{n-2}) w_{n-1}'(p^{\ddagger}_{n-2}) + w_n'(p^{\ddagger}_{n-2}) w_{n-1}(p^{\ddagger}_{n-2}). \tag{S16}$$

Dividing through by $w_{n-2}(p^{\ddagger}_{n-2}) = w_{n-1}(p^{\ddagger}_{n-2}) w_n(p^{\ddagger}_{n-2})$, this inequality is

$$\frac{w_{n-2}'(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} < \frac{w_{n-1}'(p^{\ddagger}_{n-2})}{w_{n-1}(p^{\ddagger}_{n-2})} + \frac{w_n'(p^{\ddagger}_{n-2})}{w_n(p^{\ddagger}_{n-2})}, \tag{S17}$$

a contradiction. This means that we cannot have firm $n-1$ remaining on the dance floor.

Secondly, we need to check that firm $n-1$ does not wish to return to the dance floor:

$$w_{n-1}(p) \geq (1-F_n(p))(1-F_{n-2}(p)) = \frac{w_{n-2}(p)w_n(p)}{(w_n(p^{\ddagger}_{n-2}))^2} \tag{S18}$$

This holds as an equality at $p^{\ddagger}_{n-2}$. It holds strictly for all higher p if, taking derivatives,

$$\frac{w'_{n-1}(p)}{w_{n-1}(p)} > \frac{w'_{n-2}(p)}{w_{n-2}(p)} + \frac{w'_n(p)}{w_n(p)}, \tag{S19}$$

and given that $w'_n(p) < 0$ a sufficient condition for this to hold is

$$\frac{w'_{n-1}(p)}{w_{n-1}(p)} \geq \frac{w'_{n-2}(p)}{w_{n-2}(p)} \quad \Leftrightarrow \quad c_{n-1} \leq c_{n-2}, \tag{S20}$$

which follows from differentiation of the expression for the minimum win probability. This holds because the left-hand inequality held at some price $p < p^{\ddagger}_{n-2}$. We know this because $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below as it descended at some point within $(p^{\dagger}_{n-1}, p^{\ddagger}_{n-2})$ and at the point where it crossed $w_{n-2}(p) = w_{n-1}(p)$. As an aside, this also tells us that a firm that joins the dance floor is always a firm with a higher marginal cost, and so (given that it then has a lower minimum win probability) a smaller captive audience.

Having completed these checks, we maintain our new solutions for the mixing of firms n and $n-2$. Just as before, a possibility is that $1-F_S(p) < w_i(p)$ for all $p \in [p^{\ddagger}_{n-2}, v)$ and all $i \in \{1, \ldots, n-3\}$ and, if so, we have constructed an equilibrium. Otherwise, another firm $n-3$ wishes to step in at price $p^{\ddagger}_{n-3} \in (p^{\ddagger}_{n-2}, v)$ where $1 - F_S(p^{\ddagger}_{n-3}) = w_{n-3}(p^{\ddagger}_{n-3})$. We execute another partner swap so that firm $n-3$ replaces firm $n-2$, and firm $n-2$ shifts all remaining mass to $p_{n-2} = v$.

This construction continues iteratively until we reach the upper bound v.

For generic parameter values (by which we mean that no two firms wish to join the dance floor at the same price) this construction is unique. For other knife-edge cases (including, for example, $p^{\dagger}_n = p^{\dagger}_{n-1}$ or $p^{\dagger}_{n-1} = p^{\dagger}_{n-2}$) the construction also works, but there can be multiple equilibria.

S.3. **Summary of the Equilibrium Construction for Generic Parameters.** In summary, there is a Nash equilibrium of the single-stage game in which the profit of firm $i \in \{1, \dots, n\}$ is

$$\pi_i = \lambda_i(v - c_i) + \begin{cases} (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n) & \text{if } i = n \text{, and} \\ 0 & \text{otherwise.} \end{cases} \tag{S21}$$

This means $w_i = \underline{w}_i$ as in eq. (S1) for $i < n$, while w_n is given by eq. (S6). In this equilibrium:

- Firm n plays a mixed strategy with support $[p^\dagger_{n-1}, v]$.
- Firm $n-1$ plays a mixed strategy with support $[p^\dagger_{n-1}, p^\ddagger_j]$ with $p^\ddagger_j \in (p^\dagger_{n-1}, v]$ where $p^\ddagger_j$ is the lowest such price that solves $w_j(p) = 1 - F_S(p) = w_{n-1}(p)w_n(p)$ for any $j \leq n-2$.
- No firm $i \leq n-2$ has $[p^\dagger_{n-1}, p^\ddagger_j)$ in their support.
- If $p^\ddagger_j = v$, then each $i \leq n-2$ plays the pure strategy $p_i = v$.
- If $p^\ddagger_j < v$, then j (effectively takes over the role of $n-1$ and) plays a mixed strategy with support $[p^\ddagger_j, p^\ddagger_k]$ with $p^\ddagger_k \in (p^\ddagger_j, v]$ where $p^\ddagger_k$ is the lowest such price that solves $w_k(p) = 1 - F_S(p) = w_j(p)w_n(p)/(w_n(p^\ddagger_j)^2)$ for any $k \notin \{n, n-1, j\}$.
- If $p^\ddagger_k = v$, then each $i \notin \{n, n-1, j, k\}$ plays the pure strategy $p_i = v$.
- If $p^\ddagger_k < v$, then k (effectively takes over the role of j and) the construction continues exactly as it did above for j with $p^\ddagger_j < v$.

The procedure above ends when either (i) all firms have been assigned mixed strategies or (ii) we find that for each firm yet to be assigned a mixed strategy, l, there is no $p < v$ such that $w_l(p) = 1 - F_S(p)$. In case (ii), each of those remaining firms, l, plays the pure strategy $p_l = v$.

Expressions for the CDFs for firms that play mixed strategies are recovered from eq. (S8).

This equilibrium is unique for (generic) parameters which satisfy: (i) $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$ and (ii) for any two values $p^\ddagger_i, p^\ddagger_j < v$ encountered during the iterative procedure, $p^\ddagger_i \neq p^\ddagger_j$.

Our algorithm constructs an equilibrium (for all parameter values) in which $n-1$ firms earn their captive-only profits, while firm n earns exactly $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$ more than its captive-only profit. Moreover, the equilibrium is unique for generic parameter choices, and the equilibrium profits are uniquely defined when $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.

For the remaining knife-edge cases when either $p_n^\dagger = p_{n-1}^\dagger$ or $p_{n-1}^\dagger = p_{n-2}^\dagger$ (or possibly both), we have the existence of an equilibrium with the stated profits. However, there could also be an equilibrium in which (for example) firm n earns strictly more than the stated profit. We deemed such equilibria as "pathological" in the main text and excluded them from analysis.

S.4. **Knife-Edge Cases.** Suppose that $p_n^\dagger < p_{n-1}^\dagger = p_{n-2}^\dagger$ so that the second-most aggressive firm is not uniquely defined, and further suppose (for the simplicity of exposition) that all other firms $i \in \{1, \dots, n-3\}$ choose $p_i = v$, leaving an effective three-player game between firms $i \in \{n-2, n-1, n\}$. We know that all firms other than n earn their captive-only profits, and that firm n earns Δ_n more than its captive-only profit where $\Delta_n \geq (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Suppose that this holds as a strict inequality. This implies that $\underline{p}_n > p_{n-1}^\dagger$, and so $w_n(p_{n-1}^\dagger) > 1$. Firms $n-1$ and $n-2$ must mix together down to $p_{n-1}^\dagger$, do so according to distributions $1 - F_{n-1}(p) = w_{n-2}(p)$ and $1 - F_{n-2}(p) = w_{n-1}(p)$, and so $F_S(p) = 1 - w_{n-1}(p)w_{n-2}(p)$. We know that $1 - F_S(p_{n-1}^\dagger) = 1 < w_n(p_{n-1}^\dagger)$, and that $1 - F_S(v) = 0 < w_n(v)$. Moreover, the second (strict) inequality holds even if $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. This means that $w_n(p)$ lies (strictly) above $1 - F_S(p) = w_{n-1}(p)w_{n-2}(p)$ at both the beginning and the end of the interval $[p_{n-1}^\dagger, v]$. We also know that firm n must join the dance floor at some point. This implies that there exists some $p^\ddagger$ where $w_n(p^\ddagger) = 1 - F_S(p^\ddagger) = w_{n-1}(p^\ddagger)w_{n-2}(p^\ddagger)$, which implies that $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below and then subsequently crosses from below to above within the interval $[p_{n-1}^\dagger, v]$. It is also true that $w_n(p)$ crosses $w_i(p)$ in this way for each $i \in \{n-2, n-1\}$. To proceed, we note that $w_n(p)$ is below $w_i(p)$ whenever

$$\frac{\lambda_n(v-p) + \Delta_n}{\lambda_S(p - c_n)} \leq \frac{\lambda_i(v-p)}{\lambda_S(p-c_i)} \quad \Leftrightarrow \quad \lambda_n + \frac{\Delta_n}{v-p} \leq \lambda_i \frac{p - c_n}{p - c_i}. \tag{S22}$$

The left-hand side is increasing in p. The right-hand side is decreasing in p if $c_n < c_i$. This means that $w_n(p)$ crosses $w_i(p)$ at most once from below to above. This is a contradiction. From this we conclude that if firm n has lower costs than others then there cannot be an equilibrium in which $\Delta_n > (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Equivalently, we have unique equilibrium profits. An equilibrium with $\underline{p}_n > p_{n-1}^\dagger$ must entail $c_n > c_i$, or in this case $c_n > \max\{c_{n-1}, c_{n-2})$. Firm n is the most aggressive firm, and so necessarily this also implies that $\lambda_n < \min\{\lambda_{n-1}, \lambda_{n-2}\}$.

Working with such a configuration (so that firm n has high costs but few captives), let us begin by specifying $w_n(p)$ such that $\Delta_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. We have already constructed a non-pathological equilibrium for this case. To construct a pathological equilibrium, we need

$$w'_n(p^\dagger_{n-1}) < w'_{n-1}(p^\dagger_{n-1}) + w'_{n-2}(p^\dagger_{n-1}). \tag{S23}$$

Noticing that $w_{n-2}(p^\dagger_{n-1}) = w_{n-2}(p^\dagger_{n-1}) = 1$ for this case, this says that $w_n(p)$ declines more quickly than $w_{n-1}(p)w_{n-2}(p)$ when evaluated at $p^\dagger_{n-1}$. (This inequality also stops the construction of an equilibrium in which all three firms $\{n-2, n-1, n\}$ mix as a threesome.) This means that we can raise Δ_n, so that $w_n(p^\dagger_{n-1}) > 1$, but still guarantee (so long as we don't increase Δ_n too much) that there is some larger $p^\ddagger > p^\dagger_{n-1}$ at which $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below. We then construct an equilibrium by allowing $n-1$ and $n-2$ to "dance" until $p^\ddagger$ when n joins for a "partner swap" at $\underline{p}_n = p^\ddagger$. We next illustrate with a specific example.

S.5. **Construction of Pathological Equilibria.** Consider a triopoly (so setting $n = 3$) in which two pairwise-symmetric firms have low costs but many captives, whereas the third firm has high cost and few captives. Costs satisfy $c_1 = c_2 = 0$ and $c_3 = c > 0$ while the sizes of captive audiences satisfy $\lambda_1 = \lambda_2 = \lambda_H$ and $\lambda_3 = \lambda_L$ where $\lambda_H > \lambda_L$.

We choose parameters so that firms share the same lowest undominated price $p^\dagger_1 = p^\dagger_2 = p^\dagger_3 = p^\dagger$:

$$p^\dagger = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \frac{\lambda_L v + \lambda_S c}{\lambda_L + \lambda_S} = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \quad \Leftrightarrow \quad c = \frac{(\lambda_H - \lambda_L)v}{\lambda_H + \lambda_S}. \tag{S24}$$

Henceforth when we vary the λ parameters we adjust c so that it satisfies this equation.

For this example the non-pathological equilibrium profits are captive-only for all three firms. For such profits, the required win probabilities are the minimum win probabilities. They are:

$$w_1(p) = w_2(p) = \frac{\lambda_H(v-p)}{\lambda_S p} \quad \text{and} \quad w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} \tag{S25}$$

where of course these satisfy $w_i(p^\dagger) = 1$ for all i. Also

$$w'_1(p^\dagger) = w'_2(p^\dagger) = -\frac{(\lambda_H+\lambda_S)^2}{\lambda_H\lambda_S v} \quad \text{and} \quad w'_3(p^\dagger) = -\frac{(\lambda_H+\lambda_S)(\lambda_L+\lambda_S)}{\lambda_L\lambda_S v}. \tag{S26}$$

The minimum win probability functions intersect (by construction) at $p^\dagger$. However, that function for the third firm (which has higher costs but fewer captives) declines more quickly:

$$w_3'(p^\dagger) < w_i'(p^\dagger) \text{ for } i \in \{1,2\} \;\Leftrightarrow\; -\frac{(\lambda_H+\lambda_S)(\lambda_L+\lambda_S)}{\lambda_L\lambda_S v} < -\frac{(\lambda_H+\lambda_S)^2}{\lambda_H\lambda_S v} \;\Leftrightarrow\; \lambda_L < \lambda_H. \tag{S27}$$

They key requirement to construct an equilibrium with pathological profits is that $w_3(p)$ declines more quickly than $w_1(p)w_2(p)$ when evaluated at $p^\dagger$. In this case,

$$w_3'(p^\dagger) < w_1'(p^\dagger) + w_2'(p^\dagger) \;\Leftrightarrow\; \lambda_L < \frac{\lambda_H\lambda_S}{\lambda_H + 2\lambda_S}. \tag{S28}$$

We construct an equilibrium in which firm 3 earns Δ above its captive-only profit, by setting

$$w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} + \frac{\Delta}{\lambda_S(p-c)}, \tag{S29}$$

which for $\Delta > 0$ now satisfies $w_3(p^\dagger) > 1 = w_1(p^\dagger) = w_2(p^\dagger)$. This means firm 3 does not wish "to dance" at $p^\dagger$. Instead, we construct an equilibrium in which firms 1 and 2 mix over $[p^\dagger, p^\ddagger]$ according to $1 - F_1(p) = w_2(p)$ and $1 - F_2(p) = w_1(p)$ which (from pairwise symmetry) reduces to $1 - F_i(p) = w_i(p)$ for $i \in \{1,2\}$ and $1 - F_S(p) = (w_i(p))^2$. By construction, $w_3(p) > 1 - F_S(p)$ for prices rising above $p^\dagger$. The key threshold is $p^\ddagger$ which satisfies $w_3(p^\ddagger) = 1 - F_S(p^\ddagger) = w_1(p^\ddagger)w_2(p^\ddagger)$. Explicitly, $p^\ddagger$ (which exists so long as $\Delta > 0$ is not chosen to be too large) satisfies

$$\frac{\lambda_L(v-p^\ddagger)+\Delta}{\lambda_S\left(p^\ddagger - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} = \left(\frac{\lambda_H(v-p^\ddagger)}{\lambda_S p^\ddagger}\right)^2. \tag{S30}$$

At $p^\ddagger$ there is a partner swap. Firm 2 (for example; this could be firm 3) shifts all further mass to v, which is an atom of size $w_1(p^\ddagger)$. Firms 1 and 3 then mix over the interval $[p^\ddagger, v)$ where

$$F_1(p) = 1 - \frac{w_3(p)}{w_1(p^\ddagger)} \quad \text{and} \quad F_3(p) = 1 - \frac{w_1(p)}{w_1(p^\ddagger)}. \tag{S31}$$

These firms earn their claimed equilibrium profits over this interval. The solutions satisfy $F_3(v) = 1$ (so that firm 3 does not play an atom) but $\lim_{p\uparrow v} F_1(p) < 1$ (so that firm 1 does play an atom). We need only check that firm 2 does not wish to rejoin the dance floor within this interval. We note that the probability that firm 2 wins the shoppers if it were to join is

$$(1-F_1(p))(1-F_3(p)) = \frac{w_3(p)}{w_1(p^\ddagger)}\frac{w_1(p)}{w_1(p^\ddagger)} < \frac{w_3(p^\ddagger)}{(w_1(p^\ddagger))^2}w_1(p) = w_1(p). \tag{S32}$$

S.6. **Extended Results with Innovation Choices.** In Section 2 we studied an innovation game. Propositions 5 and 6 reported our primary findings. We extend each of those in turn here. Proposition S1 provides further characterization of pure-strategy equilibria, including a condition for uniqueness. Proposition S2 covers the case of weak technological opportunities to complement Proposition 6, which dealt with the case of strong technological opportunities.

Proposition S1 (Asymmetric Innovation Equilibria). *Consider the innovation game.*

(i) There is at least one pure-strategy Nash equilibrium, and there are at most n such equilibria.

(ii) If $\arg\max_j\{V_j(z_j^L)/\lambda_j\}$ is unique, there is a unique equilibrium if λ_S is sufficiently small.

(iii) If firms are symmetric, then there are exactly n pure-strategy equilibria.

Proof of Proposition S1. We begin by re-writing the profit of firm i from eq. (13) as

$$\pi_i = \lambda_i V_i(z_i) + \lambda_S \max\left\{0, V_i(z_i) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right\}\right\} - z_i. \tag{S33}$$

As noted in the text, this is maximized by either high or low innovation choices $z_i \in \{z_i^L, z_i^H\}$ which satisfy the first-order conditions from eq. (15) and where $z_i^H > z_i^L$. In essence, the innovation game is a binary-action game where each firm chooses either high or low innovation.

Some firms will always choose low innovation. For example, any firm i where

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} \leq \max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\} \tag{S34}$$

will not choose z_i^H because it would not be the most aggressive firm (even if every other firm $j \neq i$ chooses low innovation) and so it would earn $\lambda_i V_i(z_i^H) - z_i^H < \lambda_i V_i(z_i^L) - z_i^L$. Hence we restrict to firms that do not satisfy eq. (S34), and then further restrict attention to those firms who would choose high innovation if all of their competitors choose low innovation. These firms satisfy

$$\lambda_i V_i(z_i^H) + \lambda_S \max\left\{0, V_i(z_i^H) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right\} - z_i^H \geq$$
$$\lambda_i V_i(z_i^L) + \lambda_S \max\left\{0, V_i(z_i^L) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right\} - z_i^L. \tag{S35}$$

For firms where the inequality of eq. (S34) fails (these are firms that are able to take the most aggressive position by choosing z_i^H while $j \neq i$ choose z_j^L), the inequality eq. (S35) is

$$(\lambda_i + \lambda_S)\left[V_i(z_i^H) - \max_{j \neq i}\left\{\frac{\lambda_S V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right] - z_i^H \geq \lambda_i V_i(z_i^L) + \lambda_S \max\left\{0, V_i(z_i^L) - (\lambda_i + \lambda_S)\max_{j \neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right\} - z_i^L. \quad \text{(S36)}$$

Consider the set of firms for which eq. (S35) holds. This set is non-empty. To see why, consider a firm $i \in \arg\max_{j \in \{1,\ldots,n\}} V_j(z_j^L)/(\lambda_j + \lambda_S)$. (This is a firm that would be the most aggressive (or jointly most aggressive) firm if all firms chose z_j^L.) For this firm eq. (S35) reduces to

$$(\lambda_i + \lambda_S)V_i(z_i^H) - z_i^H \geq (\lambda_i + \lambda_S)V_i(z_i^L) + \lambda_S V_i(z_i^L) - z_i^L, \quad \text{(S37)}$$

which holds strictly because z_i^H is the unique maximizer of $(\lambda_i + \lambda_S)V_i(z_i) - z_i$. Amongst the non-empty set of firms that satisfy eq. (S35), find a firm that maximizes $V_i(z_i^H)/(\lambda_i + \lambda_S)$.

We now label this firm as firm n. There is a pure-strategy Nash equilibrium of the innovation game in which $z_n = z_n^H$ and $z_j = z_j^L$ for $j \in \{1, \ldots, n-1\}$. Firm n satisfies the inequality of eq. (S35) and so does indeed wish to choose $z_n = z_n^H$. Any other firm $i \neq n$ that satisfies eq. (S35) also satisfies $V_i(z_i^H)/(\lambda_i + \lambda_H) \leq V_n(z_n^H)/(\lambda_n + \lambda_H)$, and so cannot achieve the position of the most aggressive firm by deviating to $z_i = z_i^H$. This proves claim (i).

For claim (ii), we note that $\lim_{\lambda_S \downarrow 0} z_i^H = z_i^L$. Suppose that $n = \arg\max_{j \in \{1,\ldots,n\}} V_j(z_j^L)/\lambda_j$. Then for $i \neq n$ and λ_S sufficiently small we can guarantee that $V_i(z_i^H)/(\lambda_i + \lambda_S) < V_n(z_n^L)/\lambda_n + \lambda_S)$, and so firm n must always be the most aggressive firm.

Claim (iii) is straightforward: any firm can be firm n when they are symmetric. □

Proposition S2 (The Most Aggressive Firm; Weak Opportunities). *Suppose firms have the same technology opportunities, $\lambda_1 < \cdots < \lambda_n$, and that there is a unique equilibrium (e.g. if λ_S is small). If the technological opportunity for innovation is weak then the smallest firm is the most aggressive, and amongst other firms the larger ones innovate more:*

$$p_1^\dagger < \min_{i>1}\{p_i^\dagger\} \quad \textit{and} \quad z_2^\star < \cdots < z_n^\star \quad \textit{or equivalently} \quad c_2^\star > \cdots > c_n^\star, \quad \text{(S38)}$$

but the smallest (and most aggressive) firm (which competes for shoppers) might innovate more or less than the other firms. If, additionally, λ_S is sufficiently small, then $z_1^\star < z_2^\star$, $c_1^\star > c_2^\star$, and $p_1^\dagger < \cdots < p_n^\dagger$. Furthermore, in the pricing game the two smallest (and highest cost) firms compete for shoppers, whereas other firms charge v to their captive customers.

Proof of Proposition S2. With weak technological opportunities, the key difference (from the case of strong) is that $V(z_i^L)/\lambda_i$ is decreasing in λ_i. The relevant version of eq. (20) is that for $i > 1$,

$$\lim_{\lambda_S \downarrow 0} \frac{V(z_i^H)}{\lambda_i + \lambda_S} = \frac{V(z_i^L)}{\lambda_i} < \frac{V(z_1^L)}{\lambda_1} = \lim_{\lambda_S \downarrow 0} \frac{V(z_1^L)}{\lambda_1 + \lambda_S}, \tag{S39}$$

meaning that eq. (19) fails for $i > 1$ when λ_S is sufficiently small so that only firm 1 can take the most aggressive role. Firms $i > 1$ are left serving their captives only, and so their investments (and costs) are ordered by their index. We cannot yet determine the position firm 1 takes in that ranking (the fewer total customers it serves, the less it will invest). If λ_S is sufficiently small, then a firm will serve more customers than another in total if it has more captives. When that is the case, the order of innovation intensities will follow the ranking of the size of captive audiences, such that for any two firms i and j, $\lambda_i < \lambda_j \Leftrightarrow z_i^\star < z_j^\star$. The rankings of $c_i^\star$ and $p_i^\dagger$ terms follow automatically. Because $p_1^\dagger < p_2^\dagger < p_3^\dagger$ and $\lambda_2 < \min_{i\in\{3,\ldots,n\}}\{\lambda_i\}$, claim (iv) of Proposition 2 holds and firms 1 and 2 price in mixed strategies. □

S.7. **Equilibria with a Clearinghouse.** In Section 3 we consider a model in which firms must pay a fee to reach shoppers via a clearinghouse. Here we report some basic equilibrium features.

Lemma S2 (Equilibrium with a Clearinghouse). *In an equilibrium with a clearinghouse:*

(i) Any atom in a firm's mixed strategy is placed at v or at the "no participation" decision.

(ii) The upper bound of the support for a firm that always joins the clearinghouse is v.

(iii) There are no gaps in the joint support of firms' pricing strategies below v.

(iv) At most one firm places an atom at the advertised price v.

(v) At least $n-1$ of the firms earn their captive-only profit obtained from non-participation.

Proof of Lemma S2. Claims (i) to (iii) follow from the arguments for claims (i)–(iii) of Lemma S1. Claim (iv) also follows from standard arguments: firms only advertise a price v if that can win the business of shoppers, and if two or more do so with positive probability that at least one of those firms has the incentive to undercut the atom played by the others.

For claim (v), we note that any firm that earns strictly more than its captive-only profit must always advertise a price. (Choosing not to advertise gives a firm its captive-only profit.) If two (or more) firms earn strictly more than their captive-only profits then, from claim (ii), they use a support extending up to v. At least one of those firms does not play atom at v. This means other such firms know that pricing at, or close to, v results in arbitrarily few sales to shoppers, and so their profit is arbitrarily close to the captive-only profit. This is a contradiction. $\square$

S.8. **Simultaneous Choice of Price and Technology.** In Section 2 we studied a two-stage model in which firms make technology choices (via marginal-cost-reducing process innovations) and then engage in a model-of-sales pricing game. Here we describe briefly a signal-stage version in which firms simultaneously choose both their prices and production technologies.

We use the notation from Section 2 so that z_i^L is the innovation choice of a firm i that expects to sell only to captives. This generates a captive-only profit of $\lambda_i V_i(z_i^L)$. Similarly, z_i^H is the innovation choice when expecting to sell to shoppers as well as captives. At firm i's minimum undominated price $p_i^\dagger$ it expects to serve shoppers, and so would choose z_i^H, where the corresponding marginal cost is $c_i^\dagger = v - V_i(z_i^H)$. It follows that the minimum undominated price for firm i is

$$\lambda_i V_i(z_i^L) = (\lambda_i + \lambda_S)(p_i^\dagger - v + V_i(z_i^H)) \quad \text{and so} \quad p_i^\dagger = v - V_i(z_i^H) + \frac{\lambda_i V_i(z_i^L)}{\lambda_i + \lambda_S}. \tag{S40}$$

We label firms according to their minimum undominated prices, just as in the main paper.

We can also find minimum win probabilities for a firm. Recall that the minimum win probability $\underline{w}_i(p)$ for firm i at price p is the probability of winning shoppers that makes the firm indifferent between charging p and instead earning its captive-only profit. The adjustment needed here is that we need to specify the technology choice that a firm would make if it expects to win the shoppers with probability $\underline{w}_i(p)$. Such a choice, which we label as $z_i^{(p)}$, maximizes $V_i(z_i)(\lambda_i + \underline{w}_i(p)) - z_i$.

It follows that $\underline{w}_i(p)$ and $z_i^{(p)}$ are jointly determined by solving simultaneously

$$V_i'(z_i^{(p)})(\lambda_i + \underline{w}_i(p)) = 1 \quad \text{and} \quad \lambda_i V_i(z_i^L) = (\lambda_i + \underline{w}_i(p)\lambda_S)(p - v + V_i(z_i^{(p)})). \tag{S41}$$

We can similarly construct equilibrium win probabilities. These are $w_i(p) = \underline{w}_i(p)$ for $i < n$. For the most aggressive firm n, we replace the left-hand side term $\lambda_n V_n(z_n^L)$ in the second equation displayed just above with $\lambda_n V_n(z_n^L) + (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ as before. Using these required win probabilities are repeat our equilibrium construction. Each price within the support of a firm's mixed strategy is paired with a technology (or innovation) choice.

S.9. **Relation to the Siegel (2010) Model of Contests.** Siegel (2010) studied a contest in which n players compete for $m \in \{1, \ldots, n-1\}$ (homogeneous) prizes in a single-stage game by each simultaneously choosing a "score," $s_i \geq 0$. Each of the m players with the highest scores wins one prize. For a given vector of scores $\mathbf{s} = (s_1, \ldots, s_n)$, player i's payoff is:

$$u_i(\mathbf{s}) = P_i(\mathbf{s}) v_i(s_i) - (1 - P_i(\mathbf{s})) c_i(s_i), \tag{S42}$$

where $v_i(s_i)$ is i's valuation for winning, $c_i(s_i)$ is their cost of losing such that $c_i(0) = 0$, and $P_i(\mathbf{s})$ is their probability of winning a prize (with ties broken arbitrarily).

We now rewrite the expected profit of a firm in a single-stage model of sales to show the mapping between the settings. In a model of sales there are $n \geq 2$ firms. The business of shoppers is the $m = 1$ prize, which all firms compete for.[40] Consider some profile of prices $\mathbf{p} = (p_1, \ldots, p_n)$. Instead of the highest scores winning, the lowest prices win. No firm would choose a price $p_i > v$ and so a score of zero is equivalent to a price of v. To construct the analog of eq. (S42), note that firm i's "cost of losing" when it sets v should be zero, that is, $c_i(v) = 0$. When $p_i = v$ and it loses, it makes its captive-only profit, $(v - c_i)\lambda_i$, and winning and losing are relative to that quantity. When firm i wins, it gets profit $(p_i - c_i)(\lambda_i + \lambda_S)$, which is $\lambda_S(p_i - c_i) - \lambda_i(v - p_i)$ more than its captive-only profit and so constitutes the valuation for winning. When firm i loses, it makes

[40] Notice that all firms vie for the prize. If there were customers other than captives and shoppers (i.e., with a consideration set that is not a singleton or the set of all firms), the connection between the models would break.

$(p_i - c_i)\lambda_i$, which is a loss of $(v - p_i)\lambda_i$ relative to $(v - c_i)\lambda_i$. In sum,

$$\pi_i(\mathbf{p}) = \underbrace{(v - c_i)\lambda_i}_{\text{normalization}} + P_i(\mathbf{p})\underbrace{\big(\lambda_S(p_i - c_i) - \lambda_i(v - p_i)\big)}_{\text{analog of } v_i(s_i)} - (1 - P_i(\mathbf{p}))\underbrace{(v - p_i)\lambda_i}_{\text{analog of } c_i(s_i)}. \tag{S43}$$

It is straightforward to confirm that Assumptions B1–B2 of Siegel (2010) are satisfied. Siegel indexed players by their "reach," which is the score at which the valuation for winning is zero, so that $r_i = v^{-1}(0)$. In our analysis, this corresponds to a firm's aggression, which is determined by $p_i^\dagger$ in eq. (1). Assumption B3 is equivalent to assuming $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ in our analysis.

The model of Siegel (2010) is a special case of that in Siegel (2009).[41] Among other differences, Siegel (2009) assumed each player i chooses a score $s_i \geq a_i \geq 0$. A player's "initial score" a_i is analogous to an (exogenously chosen) initial price. As we noted in the main text, these papers do not provide a full treatment for a model of sales: our Proposition 1 is covered by Siegel (2009), but our other results are not. Siegel (2010) derived equilibrium strategies for contests of m (homogeneous) prizes and $m+1$ players and so covers models of sales only in the case of duopoly.

[41]The details are described clearly by Siegel (2010, Footnote 8). Other related papers (Siegel, 2012, 2014) differ because the contest prize does not depend on a player's "score" choice (here, equivalent to the advertised price).