Honesty via Choice-Matching: Online Appendix

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Appendix

A Stochastic Relevance Under Conditional Independence and Identical Distribution

Here we show that our stochastic relevance assumption A3 is satisfied in the common setup in which there is an unknown state of the world conditioning on which respondent types are independently and identically distributed.[1]

We first derive conditions which guarantee stochastic relevance without a matching trigger. These conditions are always satisfied when the state of the world has a continuous distribution, and is generically satisfied when it has a discrete distribution. In the second step, we show that our matching trigger adds only minor additional requirements.

A.1 Stochastic relevance with respect to the state of the world

As before, let $M > 1$ be an integer and $A = \{1, ..., M\}$. We further write Δ^M for the M-dimensional simplex. We start by considering a single respondent, whose type we denote T. Suppose that Ω is a random vector taking values in Δ^M, representing the state of the world, and write

$$P(T = k \,|\, \Omega = \omega) = \omega_k \tag{A.1}$$

for all $k \in A$. In the following, we assume that Ω is a continuous random variable or a discrete random variable, and we denote by $P(\omega)$ its probability density/mass function. In the appendix, we slightly abuse notation by writing $T = k$ instead of $T^r_k = 1$ to mean that respondent r's type is k.

Definition 1. Stochastic relevance w.r.t. to Ω holds if for all different $k, j \in A$:

$$E(\Omega | T = k) \neq E(\Omega | T = j) \tag{A.2}$$

Our first result states:

[1] Note that assumption A4 is then also satisfied.

Proposition 1. *Stochastic relevance w.r.t. to* Ω *holds if and only if for all different* $k, j \in A$ *and any constant* $\lambda \in \mathbb{R}$*:*

$$P(\Omega_k = \lambda \Omega_j) < 1 \tag{A.3}$$

In words, stochastic relevance w.r.t. Ω holds if the ratio of any two state components Ω_k, Ω_j is not known with certainty ax ante. This condition is satisfied if Ω follows a continuous distribution on Δ^M because the subset of Δ^M on which $\Omega_k = \lambda \Omega_j$ has zero measure. If the distribution of Ω is discrete, then A.3 is violated only if for all ω s.t. $P(\omega) > 0$, $\omega_k = \lambda \omega_j$.

Proof. We first rewrite (A.2). W.l.o.g. take $k = 1$, $j = 2$. For any $i \in A$:

$$E(\Omega_i \,|\, T = 1) = \int\limits_{\Delta^M} P(\omega \,|\, T = 1)\, \omega_i \, d\omega = \int\limits_{\Delta^M} \omega_i \frac{P(T = 1 \,|\, \omega)}{P(T = 1)} P(\omega)\, d\omega \tag{A.4}$$

$$= \int\limits_{\Delta^M} \frac{\omega_i \omega_1}{\int_{\Delta^M} P(\omega') \, \omega'_1 d\omega'} P(\omega)\, d\omega = \frac{E(\Omega_i \Omega_1)}{E(\Omega_1)} \tag{A.5}$$

And analogously, $E(\Omega_i \,|\, T = 2) = \frac{E(\Omega_i \Omega_2)}{E(\Omega_2)}$. Thus, stochastic relevance w.r.t. Ω holds if and only if for any $i \in A$:

$$\frac{E(\Omega_i \Omega_1)}{E(\Omega_1)} \neq \frac{E(\Omega_i \Omega_2)}{E(\Omega_2)} \tag{A.6}$$

We next show that (A.3) implies (A.6) by contradiction. Suppose that (A.3) holds but that stochastic relevance w.r.t. Ω is violated, i.e. for any $i \in A$:

$$\frac{E[\Omega_i \Omega_1]}{E[\Omega_1]} = \frac{E[\Omega_i \Omega_2]}{E[\Omega_2]} \tag{A.7}$$

In particular, setting $i = 1$ and $i = 2$, respectively, (A.7) implies:

$$E\left[(\Omega_1)^2\right] = E[\Omega_1 \Omega_2] \frac{E[\Omega_1]}{E[\Omega_2]}, \qquad E\left[(\Omega_2)^2\right] = E[\Omega_1 \Omega_2] \frac{E[\Omega_2]}{E[\Omega_1]} \tag{A.8}$$

which yields

$$E\left[(\Omega_1)^2\right] E\left[(\Omega_2)^2\right] = \left(E\left[\Omega_1 \Omega_2\right]\right)^2 \tag{A.9}$$

Note that (A.9) is an instance of the Cauchy-Schwarz inequality which holds with equality if and only if there is $\lambda \in \mathbb{R}$ such that

$$P(\Omega_1 = \lambda\Omega_2) = 1 \tag{A.10}$$

Thus, we have a contradiction and (A.3) indeed implies (A.6).
Finally, it is easy to see that (A.6) implies (A.3). Note that $P(\Omega_k = \lambda\Omega_j) = 1$ implies $\lambda = \frac{E(\Omega_2)}{E(\Omega_1)}$. This immediately yields equation (A.7), in violation to (A.6). ■

Corollary 1. *Stochastic relevance w.r.t. to Ω holds if and only if for all different $k, j \in A$ there is a set $S \subset \Delta^M$ of possible values of Ω with*

$$P(\Omega \in S \,|\, T = k) \neq P(\Omega \in S \,|\, T = j) \tag{A.11}$$

In words, stochastic relevance w.r.t. to Ω holds if and only if every type disagrees with all other types about the probabilities of some states of the world.

Proof. Consider the sets

$$S_1 = \left\{\omega : \frac{\omega_k}{E[\omega_k]} > \frac{\omega_j}{E[\omega_j]}\right\}, \qquad S_2 = \left\{\omega : \frac{\omega_k}{E[\omega_k]} < \frac{\omega_j}{E[\omega_j]}\right\} \tag{A.12}$$

and recall that $P(\omega \,|\, T = k) = \frac{\omega_k}{E[\omega_k]} P(\omega)$. Thus, for $i = 1, 2$ we have

$$P(\Omega \in S_i \,|\, T = k) = \int\limits_{S_i} \frac{\omega_k}{E[\omega_k]} P(\omega)\, d\omega$$

and hence $P(\Omega \in S_i \,|\, T = k) = P(\Omega \in S_i \,|\, T = j)$ only if $P(\Omega \in S_i) = 0$. From Proposition 1, we know that stochastic relevance holds if and only if for all different $k, j \in A$ and any constant $\lambda \in \mathbb{R}$, (A.3) holds. It is straightforward to verify that (A.3) is equivalent to either $P(\Omega \in S_1) > 0$ or $P(\Omega \in S_2) > 0$ (or both). It follows that stochastic relevance holds if and only if $P(\Omega \in S_1 \,|\, T = k) \neq P(\Omega \in S_1 \,|\, T = j)$ or $P(\Omega \in S_2 \,|\, T = k) \neq P(\Omega \in S_2 \,|\, T = j)$. ■

A.2 Stochastic Relevance in a Finite Sample Distribution

Consider a finite sample of N respondents with types $T^1, ...T^N$ which satisfy conditional independence w.r.t. Ω, that is, for all $s \neq r$ and $i, k \in A$:

$$P\left(T^s = i \,|\, \Omega = \omega, T^r = k\right) = P\left(T^s = i \,|\, \Omega = \omega\right) \tag{A.13}$$

and are identically distributed, so that for all $s, r \leq N$:[2]

$$P\left(T^s = i \,|\, \Omega = \omega\right) = P\left(T^r = i \,|\, \Omega = \omega\right) = \omega_i \tag{A.14}$$

Next, recall that we defined $\bar{T}_i^{-r}$ as the frequency of respondents whose type is i in the sample excluding r.

Proposition 2. *Under conditional i.i.d, stochastic relevance w.r.t. Ω implies stochastic relevance w.r.t. $\bar{T}^{-r}$.*

Proof. W.l.o.g let $r = 1$. Then, for all $i, k \in A$:

$$E\left[\bar{T}_i^{-1} \,|\, T^1 = k\right] = \frac{1}{N-1} \sum_{s=2}^{N} Pr\left[T^s = i \,|\, T^1 = k\right] = Pr\left[T^2 = i \,|\, T^1 = k\right] \tag{A.15}$$

$$= \int\limits_{\Delta^M} P\left(T^2 = i \,|\, T^1 = k, \Omega = \omega\right) P\left(\Omega = \omega \,|\, T^1 = k\right) d\omega = E\left[\Omega_i \,|\, T^1 = k\right] \tag{A.16}$$

Thus, for given $j, k \in A$, we get that $E\left[\bar{T}^{-1} \,|\, T^1 = k\right] \neq E\left[\bar{T}^{-1} \,|\, T^1 = j\right]$ if $E\left[\Omega \,|\, T^1 = k\right] \neq E\left[\Omega \,|\, T^1 = j\right]$. ■

A.3 Sufficient Conditions For Assumption A3

We will next state a result which adapts propositions 1 and 2 to our matching trigger. As before, we write:

$$p^{r,k} := E\left[\bar{T}^{-r} \,|\, \mathcal{E}^r, T^r = k\right] \tag{A.17}$$

[2] Note that it is not restrictive that we identify the state ω with the probability ω_i. Suppose for instance that in the trial product example from section 2, honest ratings are distributed conditionally i.i.d on the quality of the product, represented by Ω. We can then simply define a new random vector $\tilde{\omega}$, s.t. $P\left(T^r = i \,|\, \Omega = \omega\right) = \tilde{\omega}_i$.

Recall that our assumption A3 states that if $j \neq k$, then:

$$p^{r,k} \neq p^{r,j}. \tag{A.18}$$

It will further prove useful to use the following identities, where $\Omega > 0$ means that $\Omega_k > 0$ for all $k \in A$.

Lemma 1. *If the types $T^1, ..., T^N$ are conditionally i.i.d w.r.t. Ω and $P(\Omega > 0) > 0$,* then for $s \neq r$ and any $i \in A$:

$$(i)\, P(T^s = i|\Omega = \omega, T^r = k, \mathcal{E}^r) = P(T^s = i|\Omega = \omega, \mathcal{E}^r) = \frac{1}{N-1} + \frac{N-1-M}{N-1}\omega_i \tag{A.19}$$

$$(ii)\, E(\bar{T}_i^{-r}|T^r = k, \mathcal{E}^r) = \frac{1}{N-1} + \frac{N-1-M}{N-1} E(\Omega_i|T^r = k, \mathcal{E}^r) \tag{A.20}$$

The first identity allows us to conclude that if $T^1, ..., T^N$ are conditionally i.i.d w.r.t. Ω, then $T^1, ..., T^N$ are also conditionally i.i.d w.r.t. to both Ω and the matching trigger. The second identity allows us to relate stochastic relevance w.r.t. Ω to stochastic relevance w.r.t. $\bar{T}^{-r}$.

Proof. In the following, w.l.o.g fix $r = k = 1$. We first prove identity *(i)*. For $s \neq 1$:

$$P(T^s = i|\Omega = \omega, T^1 = 1, \mathcal{E}^1) = \frac{P(T^s = i, \mathcal{E}^1|\Omega = \omega, T^1 = 1)}{P(\mathcal{E}^1|\Omega = \omega, T^1 = 1)} \tag{A.21}$$

Let θ^{-1} be the set of all possible $t^{-1} = (t^2, ..., t^N)$ and let $\theta_{\mathcal{E}}^{-1}$ be the set of all t^{-1} such that 1's matching trigger is in effect. We then get for the numerator:

$$P(T^s = i, \mathcal{E}^1|\Omega = \omega, T^1 = 1) = \sum_{t^{-1} \in \theta_{\mathcal{E}}^{-1}} P(T^s = i, T^{-1} = t^{-1}|\Omega = \omega, T^1 = 1) \tag{A.22}$$

$$= \sum_{\substack{t^{-1} \in \theta_{\mathcal{E}}^{-1}, \\ s.t.\, t^s = i}} P(T^{-1} = t^{-1}|\Omega = \omega, T^1 = k) = \sum_{\substack{t^{-1} \in \theta_{\mathcal{E}}^{-1}, \\ s.t.\, t^s = i}} P(T^{-1} = t^{-1}|\Omega = \omega) \tag{A.23}$$

where the first step uses that each part of the sum is zero when $t^s \neq k$ and the second step uses conditional independence. Similarly, we get $P\left(\mathcal{E}^1|\Omega=\omega,T^1=1\right)=P\left(\mathcal{E}^1|\Omega=\omega\right)$ for the denominator. Thus:

$$P(T^s=i|\Omega=\omega,T^1=1,\mathcal{E}^1)=\frac{P(T^s=k,\mathcal{E}^1|\Omega=\omega)}{P\left(\mathcal{E}^1|\Omega=\omega\right)}=P(T^s=i|\Omega=\omega,\mathcal{E}^1) \tag{A.24}$$

which is the first part of identity (i). To get the second part, note that due to conditional i.i.d. w.r.t. Ω we can rewrite

$$P(T^{-1}=t|\Omega=\omega)=\prod_{s=2}^{N}\omega_{t^s} \tag{A.25}$$

so that (A.23) yields the same expression for all $s\neq 1$. Thus, we are allowed to write

$$P(T^s=i|\Omega=\omega,\mathcal{E}^1)=\frac{1}{N-1}\sum_{s'=2}^{N}P(T^{s'}=i|\Omega=\omega,\mathcal{E}^1) \tag{A.26}$$

$$=E(\bar{T}_i^{-1}|\Omega=\omega,\mathcal{E}^1)=\frac{1}{N-1}E((N-1)\,\bar{T}_i^{-1}|\Omega=\omega,\mathcal{E}^1) \tag{A.27}$$

Let $Z=(N-1)\,\bar{T}^{-1}-1_M$, where 1_M denotes an M-dimensional vector of 1s. Note that conditioning on $\Omega=\omega$ and $\mathcal{E}^1$, Z is distributed according to a multinomial distribution, with $N-M-1$ draws and parameters $\omega_1,...,\omega_M$, such that:

$$(N-1)\;E(\bar{T}_i^{-1}|\Omega=\omega,\mathcal{E}^1)=1+E(Z_i\,|\Omega=\omega,\,Z\geq 0\,) \tag{A.28}$$

$$=1+(N-M-1)\,\omega_i \tag{A.29}$$

which also shows the second part of identity (i).

To get to identity (ii), observe that for all $i\in A$

$$E(\bar{T}_i^{-1}|T^1=k,\mathcal{E}^1)=\int_{\Delta^M}P(\,\omega|T^1=k,\mathcal{E}^1)\,E(\bar{T}_i^{-1}|T^1=k,\,\Omega=\omega,\,\mathcal{E}^1\,)d\omega \tag{A.30}$$

$$= \int_{\Delta^M} P(\,\omega|T^1 = k, \mathcal{E}^1)\,\frac{1}{N-1}\sum_{s=2}^{N} P(T^s = i\,|T^1 = k, \Omega = \omega, \mathcal{E}^1\,)d\omega \tag{A.31}$$

$$= \int_{\Delta^M} P(\,\omega|T^1 = k, \mathcal{E}^1)\left[\sum_{s=2}^{N}\frac{1}{(N-1)^2} + \frac{N-M-1}{(N-1)^2}\omega_i\right] d\omega \tag{A.32}$$

$$= \frac{1}{N-1} + \frac{N-M-1}{N-1}\,E(\omega_i|T^1 = k, \mathcal{E}^1) \tag{A.33}$$

where in (A.32) we used identity (i). This proves identity (ii) as well. ■

We will now come to the key result:

Proposition 3. *Let $N-1 > M$ and suppose that*

(i) The types $T^1,...,T^N$ are conditionally i.i.d w.r.t. Ω,

(ii) $P(\Omega > 0) > 0$

(iii) and for all different $k, j \in A$ and any constant $\lambda \in \mathbb{R}$:

$$P(\Omega_k = \lambda\Omega_j\,|\,\Omega > 0) < 1 \tag{A.34}$$

Then, assumption A3 holds.

Proof. Again fix $r = 1$. Invoking Lemma 1, we have

$$E(\bar{T}_i^{-1}|T^1 = k, \mathcal{E}^1) = \frac{1}{N-1} + \frac{N-1-M}{N-1}\,E(\Omega_i|T^1 = k, \mathcal{E}^1) \tag{A.35}$$

Thus, under assumptions *(i)* and *(ii)*, A3 holds if and only if $E(\Omega|T^1 = k, \mathcal{E}^1) \neq E(\Omega|T^1 = j, \mathcal{E}^1)$ for different j, k.

Note that $P(\Omega > 0) > 0$ implies $P(\Omega > 0\,|\,T^r = i) > 0$ for all $i \in A$. Under conditional i.i.d., this implies that for any $i \in A$:

$$P\left(\mathcal{E}^1\,|\,T^1 = i\right) > P\left(T^2 = 1, ..., T^{M+1} = M\,|\,T^1 = i\right) \tag{A.36}$$

$$= \int\limits_{\Delta^M} P\left(\omega\,|\,T^1 = i\right)\prod_{k=1}^{M}\omega_k > 0 \tag{A.37}$$

Thus, we can define $\tilde{P}(\cdot) := P\left(\cdot\,|\,\mathcal{E}^1\right)$ and $\tilde{E}(\cdot) := E\left(\cdot\,|\,\mathcal{E}^1\right)$. Lemma 1 ensures that if $T^1,...,T^N$ are independent conditional on Ω, they are independent conditional on Ω when replacing P and E by $\tilde{P}$ and $\tilde{E}$. We can thus apply proposition 1 to $\tilde{P}(\cdot)$ and $\tilde{E}(\cdot)$, which yields that assumption A3 holds if and only if for all $\lambda \in \mathbb{R}$

$$\tilde{P}(\Omega_k = \lambda\Omega_j) = P\left(\Omega_k = \lambda\Omega_j\,|\,\mathcal{E}^1\right) < 1 \tag{A.38}$$

Finally, note that

$$P\left(\Omega_k = \lambda\Omega_j\,|\,\mathcal{E}^1\right) = 1 - \int_{\Delta^M} P\left(\omega\,|\,\mathcal{E}^1\right) I_{\{\omega_k \neq \lambda\omega_j\}} d\omega \tag{A.39}$$

$$P(\Omega_k = \lambda\Omega_j\,|\,\Omega > 0) = 1 - \int_{\Delta^M} P(\omega\,|\,\Omega > 0)\, I_{\{\omega_k \neq \lambda\omega_j\}} d\omega \tag{A.40}$$

and that

$$P\left(\omega\,|\,\mathcal{E}^1\right) = P\left(\omega\,|\,\mathcal{E}^1, \Omega > 0\right) = \frac{P\left(\mathcal{E}^1\,|\,\omega, \Omega > 0\right) P(\omega\,|\,\Omega > 0)}{P(\mathcal{E}^1\,|\,\Omega > 0)} \tag{A.41}$$

For $\omega > 0$, we have that $P\left(\mathcal{E}^1\,|\,\omega, \Omega > 0\right) > 0$, so that (A.41) is strictly larger than 0 if and only if $P(\omega\,|\,\Omega > 0) > 0$. Therefore, when (A.40) does not equal 1, (A.39) does not equal 1 either. Thus, assumption *(iii)* implies (A.38) and under *(i)* and *(ii)*, assumption *(iii)* is sufficient to guarantee A3.

■

Note that in addition to the requirements of propositions 1 and 2, proposition 3 adds only the additional requirements that the sample is sufficiently large ($N - 1 > M$) and that the linear independence needed for proposition 1 holds also when restricted to $\Omega > 0$. Again, given that $N - 1 > M$, it is sufficient that Ω has a continuous distribution on Δ^M. Thus, the only cases that are excluded by our matching trigger are knife edge cases in which Ω's distribution on Δ^M is discrete, and in which, given that all types have a strictly positive (conditional) probability, the ratio of at least two types is known. Since this peculiar situation is unlikely to occur in practice, the additional restriction implied by our trigger is a very minor one.

B Proof of Proposition 2

Recall the statement of Proposition 2:

Proposition. *Let* $G = \left\langle N, A, \Omega, \{u_k\}_{k\in A}, P \right\rangle$ *be a type-separating game. Under assumptions A1-A2, any payment rule is strictly incentive compatible if it induces a game* $\left\langle N, A, \Omega, \{V_k\}_{k\in A}, P \right\rangle$ *in which on event* $\mathcal{M}^r$*:*

$$V_k(x^r, y^r, x^{-r}, y^{-r}) = \lambda u_k\left(y^r, x^{-r}, y^{-r}\right) + (1-\lambda)\,\bar{u}_k\left(x^r, x^{-r}, y^{-r}\right)$$

where $\lambda \in (0,1)$ *and* $\bar{u}$ *is the average utility value achieved by the respondents other than* r *who submit* $x_s = x_r$*, i.e.,*

$$\bar{u}_k\left(x^r, x^{-r}, y^{-r}\right) = \frac{\sum_{s\neq r} x^r \cdot x^s\, u_k\left(y^s, x^{-s}, y^{-s}\right)}{\sum_{s\neq r} x^r \cdot x^s},$$

and, on the complement of $\mathcal{M}^r$*:*

$$V_k(x^r, y^r, x^{-r}, y^{-r}) = 0.$$

Proof. As in the proof of Proposition 1, observe that r cannot influence the choice-matching trigger and should thus condition his expected payoffs on being choice-matched. Since G is type separating, condition (*iii*) from definition 4 guarantees that the payment rule is strictly incentive compatible in y. Consider then the difference in the expected score for respondent r between reporting t^r honestly and deviating by making some dishonest report x^r with $x_i^r = 1$:

$$P(\mathcal{E}^r \mid t_k^r = 1) \times (1-\lambda)\, E\left[\bar{u}_k\left(t^r, x^{-r}, y^{-r}\right) - \bar{u}_k\left(x^r, x^{-r}, y^{-r}\right) \mid T^r = k, \mathcal{E}^r\right]$$

Due to the construction of $\bar{u}\left(x^r, x^{-r}, y^{-r}\right)$:

$$\bar{u}_k\left(t^r, x^{-r}, y^{-r}\right) - \bar{u}_k\left(x^r, x^{-r}, y^{-r}\right) = u_k\left(y^{*k}, x^{-r}, y^{-r}\right) - u_k\left(y^{*i}, x^{-r}, y^{-r}\right)$$

for $i \neq k$. Again, using condition (*iii*) from definition 4, we have that for r's conditional expectation of this payoff difference:

$$E\left[u_k\left(y^{*k}, x^{-r}, y^{-r}\right) - u_k\left(y^{*i}, x^{-r}, y^{-r}\right) \mid T^r = k, \mathcal{E}^r\right] > 0$$

which gives the required result. ■