Policy Options at the Zero Lower Bound when Foresight is Limited

Michael Woodford and Yinxi Xie

Online Appendix – Not for Publication

A Model Settings

To study the policy options at the Zero Lower Bound (ZLB) when planning horizon is finite, we relax the assumption of perfectly model-consistent expectations in a particular way proposed in Woodford (2018). More specifically, decision makers are assumed to be capable of explicit forward planning some finite distance into the future. Within planning horizon h, agents deduce the consequences of a finite sequence of possible actions, using known structure of the environment which includes any newly announced government's policies. Beyond the limited planning horizon, it is too costly to continue searching all possible branches of the decision tree, and therefore, agents evaluate the states at the end of planning horizon using a value function that has been learned from prior experiences, i.e., $v(B)$ for households and $\tilde{v}(P^f/P)$ for firms, which necessarily takes into account only a coarse description of the state.

Since the crisis period is temporary, and we focus on temporary policy changes in response to the disturbances during crisis periods, unless otherwise stated, we assume that the value functions $v(B)$ and $\tilde{v}(P^f/P)$ are given. In other words, they do not evolve in response to experiences during crisis, but instead are fixed at particular values, the functions that would be appropriate to the "normal steady state" that is assumed to have existed before the exogenous disturbance occurs. On the one hand, we assume that duration of crisis state is short enough to allow us to abstract from adjustment of value functions owing to learning about the new regime under crisis. On the other hand, the assumption of non-evolving value function during crisis can be regarded as a limiting case of the full model, i.e., the limit in which learning is very slow. It will simplify the analysis and explanation of the results, and since we are only interested in temporary disturbances and temporary policy changes in response to those disturbances, this is not necessarily a bad assumption.

Following Woodford (2018), by log-linearization, for decision makers with planning horizon $h \geq 1$, we have

$$y_t^h = E_t^h y_{t+1}^{h-1} - \sigma[\hat{\imath}_t - \hat{r}_t^n - E_t^h \pi_{t+1}^{h-1}] \tag{A.1}$$

$$\pi_t^h = \kappa y_t^h + \beta E_t^h \pi_{t+1}^{h-1} \tag{A.2}$$

Here, y_t^h and π_t^h are real expenditure by households with planning horizon h and the rate of price increase by firms with planning horizon h for a particular fundamental state at time t. Note that y_t is measured as a log-deviation from the steady-state output level consistent with inflation target π^*, and π_t is measured as a deviation from the target π^*. $\hat{r}_t^n$ denotes the deviation from the steady state real return on safe assets, and $\hat{\imath}_t$ denotes the current policy choice, which is also written in terms of a deviation from the steady state. We assume that all agents have a correct awareness of the current fundamental state $\hat{r}_t^n$ and of the current policy choice $\hat{\imath}_t$. $E_t^h[\cdot]$ means the expected value of future variables under the beliefs of horizon-h planners, in particular, under the assumption $E_t^h[\tilde{P}_t] = \tilde{P}_{t-1} + \Pi_t^h$ about what the lagged "price level gap" will be in the next period. Forward planners correctly understand how y_{t+1}^{h-1} and π_{t+1}^{h-1} will depend on $\tilde{P}_t$; but in predicting $\tilde{P}_t$, they assume that all price-setters will behave like them, so that the inflation rate will be $\pi_t = \pi_t^h$. For the case of $h = 0$, the aggregate demand IS curve and New Keynesian Phillips curve will depend on the policy regime, which will be discussed later in the analysis of specific alternative monetary and fiscal policies.

We consider the effects of alternative monetary and fiscal policies under the following scenario: prior to date $t = 0$, we suppose that the economy has for a long time been in a regime under which there are no financial frictions (hence, natural rate of interest rate $r_t^n = r^* > 0$), government purchases are constant and government's budget is balanced in each period, and the inflation target π^* has been consistently achieved (the ZLB is no obstacle to this). As a result, households and firms have learned the value functions that are appropriate to such a regime. At time $t = 0$, unexpected shock on fundamental occurs, creating a wedge $\Delta > 0$ between the return on safe assets (balances held at the central bank) and other assets.

Moe specifically about the fundamentals $\{\hat{r}_t^n\}$, it follows an exogenous process, i.e., a two-state Markov chain. The two states are $r_t^n = r^* > 0$ in "normal" state and $r_t^n = \underline{r} < 0$ in "crisis" state, implying that the financial wedge equals $\Delta = r^* - \underline{r}$. We suppose that, at date $t = 0$, the economy enters the "crisis" state, and thus $r_0^n = \underline{r}$ after a long time in the steady state with $r_t^n = r^*$ and inflation equal to π^*. Once in the crisis state, there is a probability $0 < \delta < 1$ in each period of continuing to be in the "crisis" state again in the following period, and a probability $1 - \delta$ of reverting to the "normal" state. For simplicity,

Table 1: Calibrated Parameter Values

	Value
Subject discount factor	$\beta = 0.997$
Response of inflation to output gap in Phillips curve	$\kappa = 0.00859$
Risk aversion	$\sigma = 0.862$
Fiscal multiplier in flexible price	$\Gamma = 0.425$
Natural rate in "crisis" state	$\underline{r} = -0.010$
Probability of staying in "crisis" state	$\delta = 0.903$
Inflation target	$\pi^* = 0.005$

this probability of exit is independent of the length of time already spent in the "crisis" state. Once the economy returns to the "normal" state, it is expected to remain there forever, i.e., the financial wedge becomes zero again and is expected to be zero thereafter. The variable $\hat{r}_t^n$ in $(A.1)$ is written in terms of deviations from the steady state value in the "normal" state, i.e., $\hat{r}_t^n = 0$ in the "normal" state while $\hat{r}_t^n = \underline{r} - r^* \equiv -\Delta < 0$ in the "crisis" state.

Since $\hat{\imath}_t$ is also written in terms of a deviation from the "normal" steady state in which the constant nominal interest is $r^* + \pi^*$, the zero-lower-bound constraint requires that

$$\hat{\imath}_t \geq -(r^* + \pi^*) \tag{A.3}$$

in all periods. Note that we assume $r^* > 0$, $\pi^* > 0$.

A.1 Calibration

In order to compare with the results under rational expectation equilibrium (REE), we calibrate the parameters to be the same as those in REE analysis, as in Woodford (2011). The parameters are chosen to fit the size of the contraction experienced by the US economy during the Great Depression. For a quarter model, we set the subject discount factor $\beta = 0.997$, the response of inflation to output gap in Phillips curve $\kappa = 0.00859$, the inverse of elasticity of intertemporal substitution $\sigma = 0.862$ (i.e., the relative risk aversion), and the fiscal multiplier in flexible price $\Gamma = 0.425$. The shock required to account for the size of the contraction during the Great Depression is $\underline{r} = -0.010$ and the probability of staying in "crisis" state is $\delta = 0.903$. Note that the natural rate in the "normal" time steady state is given by $r^* = 1/\beta - 1$. In addition, we adopt a 2% annual inflation target, i.e., $\pi^* = 0.005$. The calibrated parameter values are summarized in Table 1.

B Monetary Policy: Inflation Targeting

In this section, we study what should happen when the crisis occurs, if there is no change in either fiscal or monetary policy, and the monetary policy is specified by a strict inflation target: inflation rate π^* is maintained as long as consistent with the ZLB. More specifically, suppose that the central bank achieves its inflation target π^* whenever it is consistent with $(A.3)$, and sets $\hat{\imath}_t$ as lower as possible otherwise. Its policy rule therefore requires that

$$\pi_t \leq 0 \tag{B.1}$$

at all times, and that either $(A.3)$ or $(B.1)$ must hold with equality in each period.

Because of the Markovian form of the equilibrium relations including the specified policy rule $(B.1)$, regardless of planning horizon h, the solution will be Markovian as well. That is, the value of each of the variables $\{y_t^k, \pi_t^k\}$, as well as the value of $\hat{\imath}_t$, will depend only on which of the two states the fundamentals are in at date t.

Solution once "normal" state is reached: in this case, the central bank can ensure $\pi_t = 0$ forever. Conditions $(A.1)$-$(A.2)$ for the types with horizon $h = 0$ require

$$y_t^0 = -\sigma\hat{\imath}_t, \pi_t^0 = \kappa y_t^0 \tag{B.2}$$

which imply constant values $y_t^0 = \bar{y}^0$, $\pi_t^0 = \bar{\pi}^0$ as long as $\hat{\imath}_t = \bar{\imath}$ is constant. Conditions $(A.1)$-$(A.2)$ for horizon $h = 1$ then imply

$$y_t^1 = \bar{y}^0 - \sigma\hat{\imath}_t + \sigma\bar{\pi}^0, \pi_t^1 = \kappa y_t^1 + \beta\bar{\pi}^0$$

so that we also have constant values $y_t^1 = \bar{y}^1$, $\pi_t^1 = \bar{\pi}^1$. We can proceed recursively to solve for $y_t^h = \bar{y}^h$, $\pi_t^h = \bar{\pi}^h$ for any $h \geq 0$. The constant value of $\hat{\imath}_t$ required to maintain $\pi_t = 0$ is $\hat{\imath}_t = 0$; under this assumption, the above equations imply $\bar{y}_t^h = 0$, $\bar{\pi}^h = 0$ for all $h \geq 0$. Hence, $\pi_t = 0$ in each period, regardless of the assumed distribution of planning horizons.

Solution in the "crisis" state: conditions $(A.1)$-$(A.2)$ for horizon $h = 0$ now reduce to $y_t^0 = -\sigma[\hat{\imath}_t + \Delta]$, $\pi_t^0 = \kappa y_t^0$, and again imply constant values $y_t^0 = \underline{y}^0$, $\pi_t^0 = \underline{\pi}^0$ as long as $\hat{\imath}_t = \underline{\hat{\imath}}$ is constant. Conditions $(A.1)$-$(A.2)$ for horizon $h = 1$ then imply

$$y_t^1 = [\delta\underline{y}^0 + (1-\delta)\bar{y}^0] - \sigma[\underline{\hat{\imath}} + \Delta] + \sigma[\delta\underline{\pi}^0 + (1-\delta)\bar{\pi}^0]$$

$$\pi_t^1 = \kappa y_t^1 + \beta[\delta\underline{\pi}^0 + (1-\delta)\bar{\pi}^0]$$

so that we also have constant values $y_t^1 = \underline{y}^1$, $\pi_t^1 = \underline{\pi}^1$. We can proceed recursively in this way to solve for $y_t^h = \underline{y}^h$, $\pi_t^h = \underline{\pi}^h$ for any $h \geq 0$, as fundamentals of the assumed constant interest rate $\hat{\underline{\imath}}$. If it were possible to set $\hat{\underline{\imath}} = -\Delta$, the above equations would imply $y_t^h = 0$, $\pi_t^h = 0$ for all $h \geq 0$, and the inflation target would be achieved by this policy. We assume, however, that $\Delta > r^* + \pi^*$, so that this level of interest rate would violate $(A.3)$. The only possible solution is therefore the one with $\hat{\underline{\imath}} = -(r^* + \pi^*)$, i.e., the lower bound, in the "crisis" state. We can then solve the following recursive system of equations for $\{\underline{y}^h, \underline{\pi}^h\}$:

$$\begin{cases} \underline{y}^0 = -\sigma\tilde{\Delta} & \underline{\pi}^0 = \kappa\underline{y}^0 \\ \underline{y}^1 = \delta\underline{y}^0 - \sigma\tilde{\Delta} + \sigma\delta\underline{\pi}^0 & \underline{\pi}^1 = \kappa\underline{y}^1 + \beta\delta\underline{\pi}^0 \\ \underline{y}^2 = \delta\underline{y}^1 - \sigma\tilde{\Delta} + \sigma\delta\underline{\pi}^1 & \underline{\pi}^2 = \kappa\underline{y}^2 + \beta\delta\underline{\pi}^1 \\ \dots & \dots \end{cases} \tag{B.3}$$

where $\tilde{\Delta} \equiv \Delta - (r^* + \pi^*) > 0$.

The solution for actual aggregate output and inflation in the crisis state then depends on what we assume about the distribution of forecast horizons in the population. If everyone has the same planning horizon h, then $y_t = \underline{y}^h$, $\pi_t = \underline{\pi}^h$ in the "crisis" state. Given the calibration in Section A.1, Figure 1 shows the constant-level of output and inflation in "crisis" state as a function of planning horizon h, where the unit in h indicates one quarter. With finite planning horizons, effects of the shock are smaller than under rational expectation equilibrium (REE) analysis such as Eggertsson and Woodford (2003), Eggertsson (2010), and as agents are more short-foresight, the contraction is less severe.

If instead we assume a distribution of population fractions $\{\omega_j\}_{j=0}^{\infty}$, then $y_t = \Sigma_j \omega_j \underline{y}^j$, $\pi_t = \Sigma_j \omega_j \underline{\pi}^j$. A simple case to solve is the case of an exponential distribution of forecast horizons, $\omega_j = (1-\rho)\rho^j$ for $j \geq 0$. In this case, relations $B.3$ can be summed to yield

$$y^{crisis} = \delta\rho y^{crisis} - \sigma\tilde{\Delta} + \sigma\delta\rho\pi^{crisis}$$

$$\pi^{crisis} = \kappa y^{crisis} + \beta\delta\rho\pi^{crisis}$$

This has a unique solution

$$\pi^{crisis} = \frac{-\kappa\sigma\tilde{\Delta}}{[(1-\rho\delta)(1-\beta\delta\rho) - \kappa\sigma\delta\rho]} < 0$$

$$y^{crisis} = \frac{-\sigma(1-\beta\delta\rho)\tilde{\Delta}}{[(1-\rho\delta)(1-\beta\delta\rho) - \kappa\sigma\delta\rho]} < 0$$

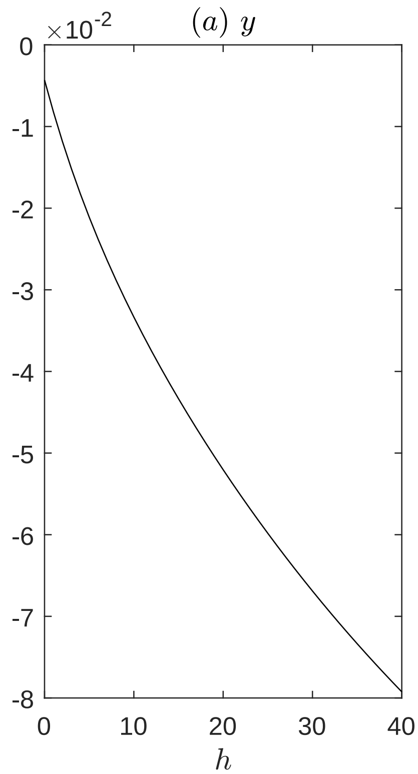

Figure 1: Output and Inflation in "Crisis" State as a Function of Planning Horizon

under the assumption that

$$(1-\rho\delta)(1-\beta\delta\rho)-\kappa\sigma\delta\rho>0 \tag{B.4}$$

Note that if $(B.4)$ does not hold, the infinite sums $\Sigma_j\omega_j\underline{y}^j$, $\Sigma_j\omega_j\underline{\pi}^j$ diverge. But this possibility is not of realistic interest, since surely there should be some finite upper bound beyond which $\omega_j=0$; the assumption of an exponential distribution is an approximation for the sake of convenience in the algebraic calculation.

C Government Purchases with Inflation Targeting

Since the monetary policy may be constrained by the ZLB during the crisis periods, in this section, we study real government purchases as a type of "fiscal stimulus": government purchases are increased by a constant amount as along as "crisis" state persists, and return to normal level when the economy reverts back to "normal" state. More specifically, we assume that government purchases G_t follow a two-state Markov process, i.e., $G_t=0$ in the "normal" state and $G_t=G>0$ in the "crisis" state. Hence, the government purchases g_t as a percentage of steady-state output in "normal" state is given by $g_t=0$ in the "normal" state and $g_t = G/\bar{Y} > 0$ in the "crisis" state, where $\bar{Y}$ is the steady-state output level.[1]

Suppose the fundamentals $\{\hat{r}^n_t\}$ follow the same exogenous process as in Section B, and the monetary policy is also the same as specified in Section B (i.e., inflation targeting). For simplicity, we assume a balanced-budget policy, i.e., no change in the time-path of real government debt b_t.

Following Woodford (2011, 2018), by log-linearization, for decision makers with planning horizon $h \geq 1$, we have

$$y^h_t - g_t = E_t[y^{h-1}_{t+1} - g_{t+1}] - \sigma[\hat{\imath}_t - \hat{r}^n_t - E_t \pi^{h-1}_{t+1}] \tag{C.1}$$

$$\pi^h_t = \kappa[y^h_t - \Gamma g_t] + \beta E_t \pi^{h-1}_{t+1} \tag{C.2}$$

with

$$y^0_t - g_t = -\sigma[\hat{\imath}_t - \hat{r}^n_t]$$

$$\pi^0_t = \kappa[y^0_t - \Gamma g_t]$$

where $\Gamma = \eta_u/(\eta_u + \eta_w) < 1$. Note that $\eta_u = -\bar{Y}u''/u' > 0$ is the negative elasticity of u' and $\eta_w = \bar{Y}\tilde{w}''/\tilde{w}'$ is the elasticity of $\tilde{w}$ with respect to increases in Y.[2]

C.1 The Rational Expectation Equilibrium (REE)

Solution once "normal" state is reached: $\pi_t = 0$ and $y_t = 0$ for all periods, and target inflation rate is achieved and is expected to be achieved forever after. This is the result with rational expectations. No further discussion is needed for this case.

Solution in the "crisis" state: under rational expectations, $\pi_t = \underline{\pi}^{RE}$, $y_t = \underline{y}^{RE}$ in all periods, where $(\underline{\pi}^{RE}, \underline{y}^{RE})$ are functions of g that satisfy

$$\underline{y}^{RE} - g = \delta(\underline{y}^{RE} - g) - \sigma(\hat{\underline{\imath}}^{RE} + \Delta) + \sigma\delta\underline{\pi}^{RE}$$

$$\Rightarrow (1-\delta)(\underline{y}^{RE} - g) = -\sigma(\hat{\underline{\imath}}^{RE} + \Delta) + \sigma\delta\underline{\pi}^{RE}$$

and

$$\underline{\pi}^{RE} = \kappa(\underline{y}^{RE} - \Gamma g) + \beta\delta\underline{\pi}^{RE}$$

$$\Rightarrow (1-\beta\delta)\underline{\pi}^{RE} = \kappa(\underline{y}^{RE} - \Gamma g)$$

[1] At least in the case that G is not very large compared with $\bar{Y}$, the log-linear approximation is accurate.

[2] Similar to the notation in Woodford (2018), the period utility of household i is defined as $u(C^i_t) - w(H^i_t)$, where C^i_t is the quantity consumed in period t and H^i_t is hours of labor supplied in period t. As usual, $u(\cdot)$ is an increasing, strictly concave function, and $w(\cdot)$ is an increasing, convex function. Note that $\tilde{w}(Y) = w(f^{-1}(Y))$ is the disutility to the household of supplying a quantity of output Y, and f is the production technology.

In order to achieve the inflation target, it requires to have $\underline{\pi}^{RE} = 0$, which implies $\underline{y}^{RE} = \Gamma g$. In this case, we have

$$\sigma(\underline{\hat{\imath}}^{RE} + \Delta) = -(1-\delta)(\underline{y}^{RE} - g) + \sigma\delta\underline{\pi}^{RE}$$

$$= (1-\delta)(1-\Gamma)g$$

which is feasible (to be consistent with $\underline{\hat{\imath}}^{RE} \geq -(r^* + \pi^*) \Longleftrightarrow \underline{\hat{\imath}}^{RE} + \Delta \geq \tilde{\Delta}$) if and only if $g \geq \bar{g}^{RE}$, where $\bar{g}^{RE}$ is defined by

$$\sigma\tilde{\Delta} = (1-\delta)(1-\Gamma)\bar{g}^{RE}$$

If instead $g < \bar{g}^{RE}$, the ZLB binds, and $(\underline{\pi}^{RE}, \underline{y}^{RE})$ are given by

$$(1-\delta)(\underline{y}^{RE} - g) = -\sigma\tilde{\Delta} + \sigma\delta\underline{\pi}^{RE}$$

$$(1-\beta\delta)\underline{\pi}^{RE} = \kappa(\underline{y}^{RE} - \Gamma g)$$

which have a solution[3]

$$\underline{\pi}^{RE} = \frac{\kappa[(1-\delta)(1-\Gamma)g - \sigma\tilde{\Delta}]}{(1-\beta\delta)(1-\delta) - \sigma\delta\kappa} < 0$$

so that the inflation target cannot be achieved even with $\hat{\imath}$ at its lower bound.

C.2 The Equilibrium with Finite Planning Horizon

For agents with any finite horizon $h \geq 0$, the solution once the "normal" state is reached is the same with that of REE case, and thus no further discussion needed. Instead, we focus on the solution in the "crisis" state.

If all have horizon $h = 0$: then while still in the crisis state, $(\underline{\pi}^0, \underline{y}^0)$ are functions of g that satisfy

$$\underline{y}^0 - g = -\sigma(\underline{\hat{\imath}}^0 + \Delta)$$

$$\underline{\pi}^0 = \kappa(\underline{y}^0 - \Gamma g)$$

In order to achieve the inflation target, it requires to have $\underline{\pi}^0 = 0 \Rightarrow \underline{y}^0 = \Gamma\hat{G}$. In this case, we have

$$\sigma(\underline{\hat{\imath}}^0 + \Delta) = -(\underline{y}^0 - g) = (1-\Gamma)g$$

[3]We assume that $\delta\sigma\kappa < (1-\beta\delta)(1-\delta)$, so that the REE solution exists.

which implies $\underline{\hat{\imath}}^0 = \frac{1-\Gamma}{\sigma}g - \Delta$, and it is feasible if and only if $g \geq \bar{g}^0$, where $\bar{g}^0$ is defined by

$$(1-\Gamma)\bar{g}^0 = \sigma\tilde{\Delta}$$

and satisfies $0 < \bar{g}^0 < \bar{g}^{RE}$.

If instead $g < \bar{g}^0$, the ZLB binds and $(\underline{\pi}^0, \underline{y}^0)$ are given by

$$\underline{\pi}^0 = \kappa[(1-\Gamma)g - \sigma\tilde{\Delta}] < 0 \tag{C.3}$$

$$\underline{y}^0 = g - \sigma\tilde{\Delta} \tag{C.4}$$

For any finite horizon $h \geq 1$: then while in the crisis state, $(\underline{\pi}^h, \underline{y}^h)$ are functions of g that satisfy

$$\underline{y}^h - g = \delta[\underline{y}^{h-1} - g] - \sigma(\underline{\hat{\imath}}^h + \Delta) + \sigma\delta\underline{\pi}^{h-1}$$

$$\underline{\pi}^h = \kappa[\underline{y}^h - \Gamma\hat{G}] + \beta\delta\underline{\pi}^{h-1}$$

1. If $g \geq \bar{g}^{RE}$: then the inflation target is achieved, and is expected to be achieved over the rest of the planning horizon. Consequently, we have

$$\underline{\pi}^h = \underline{\pi}^{h-1} = 0 \Rightarrow \underline{y}^h = \Gamma\hat{G}$$

   - Proof: Proceed by induction. Suppose that we have already shown that

$$\underline{\pi}^j = 0, \underline{y}^j = \Gamma\hat{G}$$

   for all $j \leq h-1$. We wish to show that it is also true for $j = h$. Since $\underline{\pi}^{j-1} = 0$, achievement of $\underline{\pi}^h = 0$ would require $\underline{y}^h = \Gamma\hat{G}$. This would then require that

$$-\sigma(\underline{\hat{\imath}}^h + \Delta) = \underline{y}^h - g - \delta(\underline{y}^{h-1} - g) - \sigma\delta\underline{\pi}^{h-1}$$

$$= -(1-\delta)(1-\Gamma)g$$

$$\Rightarrow \underline{\hat{\imath}}^h = \frac{(1-\delta)(1-\Gamma)}{\sigma}g - \Delta$$

   But this is consistent with the ZLB as long as $g \geq \bar{g}^{RE}$. Q.E.D.

2. If $\bar{g}^0 \leq g < \bar{g}^{RE}$: then the ZLB binds, and is expected to continue to bind for h periods, but is not expected to bind in the final period of the finite-horizon planning exercise.

We have already shown that if $h = 0$, the ZLB does not bind in this case, i.e.,

$$\underline{\pi}^0 = 0, \underline{y}^0 = \Gamma g \tag{C.5}$$

Under the above conjecture, $(\underline{\pi}^h, \underline{y}^h)$ for any $h \geq 1$ must satisfy

$$\underline{y}^h - g = \delta[\underline{y}^{h-1} - g] - \sigma\tilde{\Delta} + \sigma\delta\underline{\pi}^{h-1} \tag{C.6a}$$

$$\underline{\pi}^h = \kappa[\underline{y}^h - \Gamma g] + \beta\delta\underline{\pi}^{h-1} \tag{C.6b}$$

The system (C.6a)-(C.6b) can be solved recursively, starting from the initial values (C.5). It needs to shown that the solution to this system satisfies $\underline{\pi}^h < 0$ for all $h \geq 1$, thus verifying that the ZLB binds.

- Proof: again proceed by induction. Suppose that we have already shown that

  $$\underline{\pi}^j \leq 0, \underline{y}^j \leq \Gamma g \tag{C.7}$$

  for all $j \leq h - 1$. Then, (C.6a) implies that

  $$\underline{y}^h - \Gamma g = (1 - \Gamma)g + \delta[\underline{y}^{h-1} - g] - \sigma\tilde{\Delta} + \sigma\delta\underline{\pi}^{h-1}$$

  $$\leq (1 - \delta)(1 - \Gamma)g - \sigma\tilde{\Delta} < 0$$

  and (C.6b) then implies that

  $$\underline{\pi}^h \leq \kappa(\underline{y}^h - \Gamma g) < 0$$

  Since we have already established in (C.5) that (C.7) holds when $j = 0$, we get $\underline{\pi}^h \leq 0, \underline{y}^h \leq \Gamma g$ for all $h \geq 1$. Q.E.D.

3. If $g < \bar{g}^0$: then the ZLB binds, and is expected to continue to bind over the entire planning horizon. Because the ZLB binds even when $h = 0$, we have $(\underline{\pi}^0, \underline{y}^0)$ given by (C.3)-(C.4). For any $h \geq 1$, $(\underline{\pi}^h, \underline{y}^h)$ must again satisfy (C.6a)-(C.6b). This system can also be solved recursively, starting from the initial values (C.3)-(C.4). It needs to be shown that the solution satisfies $\underline{\pi}^h < 0$ for all $h \geq 0$, thus verifying that the ZLB binds.

   - Proof: proceed by induction. In fact, the proof for the case of $\bar{g}^0 \leq g < \bar{g}^{RE}$ works again. If (C.7) holds for all $j \leq h - 1$, then (C.6a)-(C.6b) imply that

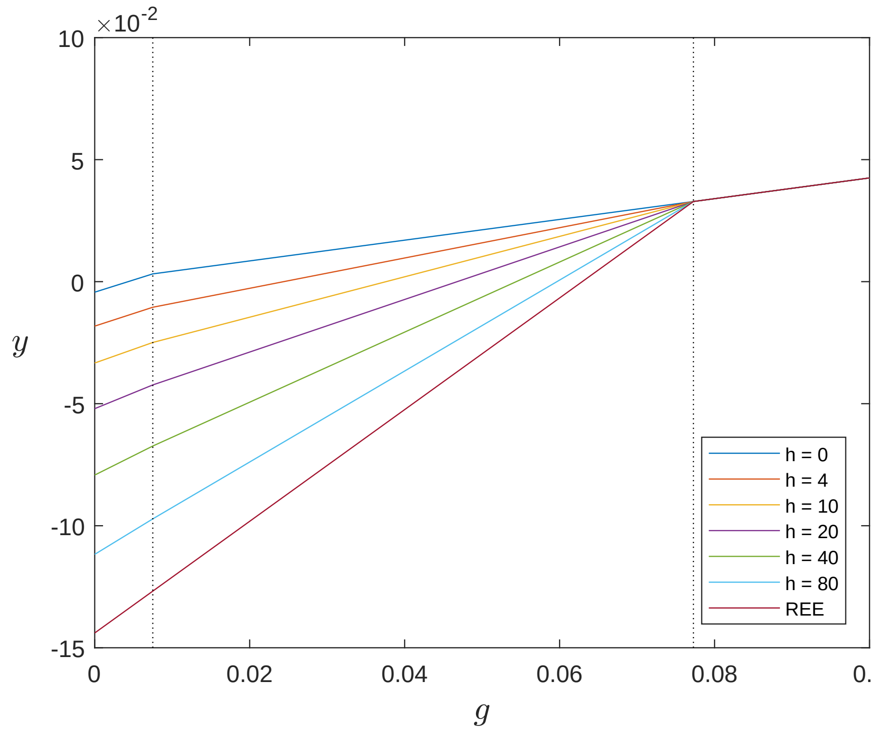

Figure 2: Output as a Function of Government Purchases during "Crisis" Periods

> $\underline{\pi}^h \leq 0, \underline{y}^h \leq \Gamma g$. Since we have already shown, in (C.3)-(C.4), that (C.7) holds when $j = 0$, by induction, $\underline{\pi}^h \leq 0$, $\underline{y}^h < \Gamma g$ for all $h \geq 0$, and the ZLB does indeed binds for all $h \geq 0$. Q.E.D.

The solution for actual aggregate output and inflation in the crisis state then depends on what we assume about the distribution of forecast horizons in the population. If everyone has the same horizon h, then $y_t = \underline{y}^h$, $\pi_t = \underline{\pi}^h$ in the "crisis" state. Given the calibration in Section A.1, Figure 2 shows the output response to government purchases in "crisis" state with different planning horizons. Through the intercept on y-axis, it also illustrates how the planning horizon affects the output contraction induced by the negative fundamentals. The two dotted lines indicates the thresholds of $\bar{g}^0$ and $\bar{g}^{RE}$, respectively.

Foresight is crucial to the size of fiscal multiplier. REE analysis by Eggertsson (2010), Christiano et al. (2011), and Woodford (2011) find a multiplier well above 1 if the crisis state is expected to be consistent. But with finite planning horizons, fiscal multipliers can be much smaller. When the government purchase is small (i.e., $g < \bar{g}^0$), Figure 3 shows the fiscal multiplier with respect to different planning horizons, and if the horizon h is short, the initial multiplier can be as low as 1.[4] In contrast with REE analysis, when h is finite, initial

[4] While the tick markers indicates the planning horizons in quarters, the planning horizons are shown on a log scale on x-axis.

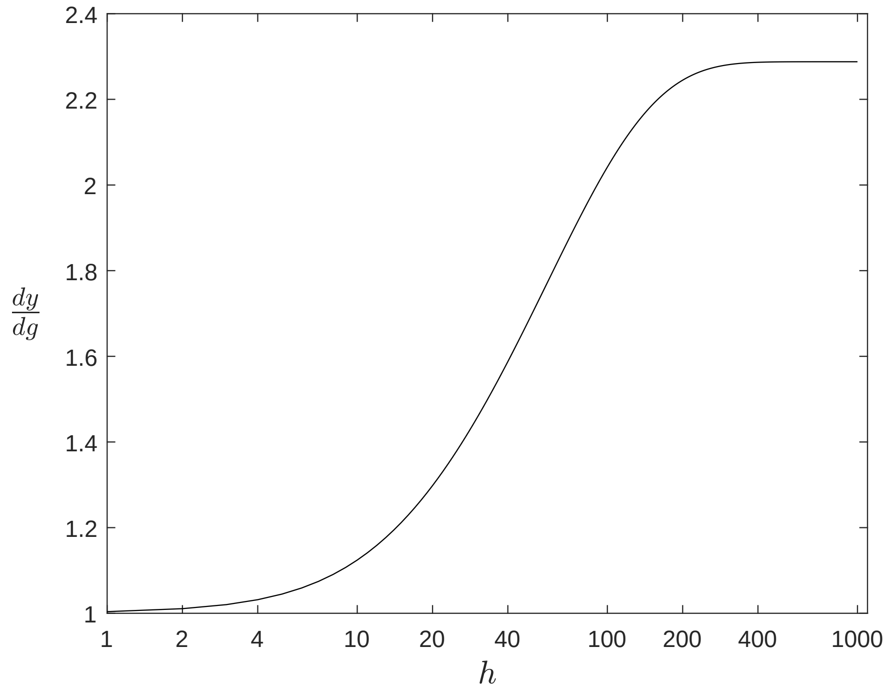

Figure 3: Fiscal Multiplier with Small Government Purchases

multiplier does not continue to apply for any very substantial increase in g, i.e., $\bar{g}^0 \leq g < \bar{g}^{RE}$. With a substantial increase in g, Figure 4 shows the fiscal multiplier with respect to different planning horizons, and if horizon h is short, the initial multiplier can be even less than 1.

In the numerical calibration, increasing g by even 1 percent of GDP is enough to make the fiscal multiplier fall to lower level. Actually, this depends on assuming a quarterly model used for forward planning – if periods are shorter, the increase in g that causes the multiplier to fall is even smaller – zero in the continuous-time limit. Therefore, the relevant multiplier is really the one associated with the intermediate region in 2, where output gap is eliminated for the case of $h = 0$, and as illustrated in Figure 4.

D Lump-sum Transfer Financed by Government Debt with Inflation Targeting

Allowing for shorter planning horizons increases the predicted effectiveness of "fiscal stimulus" programs in one respect: it provides a reason for deficit-financed government spending to be more stimulative. Ricardian equivalence will not hold with finite planning horizons, if we assume that people have not already learned to condition their value functions on the size of public debt. In this section, we consider a government lump-sum transfer financed by government debt with inflation targeting as specified in Section B.

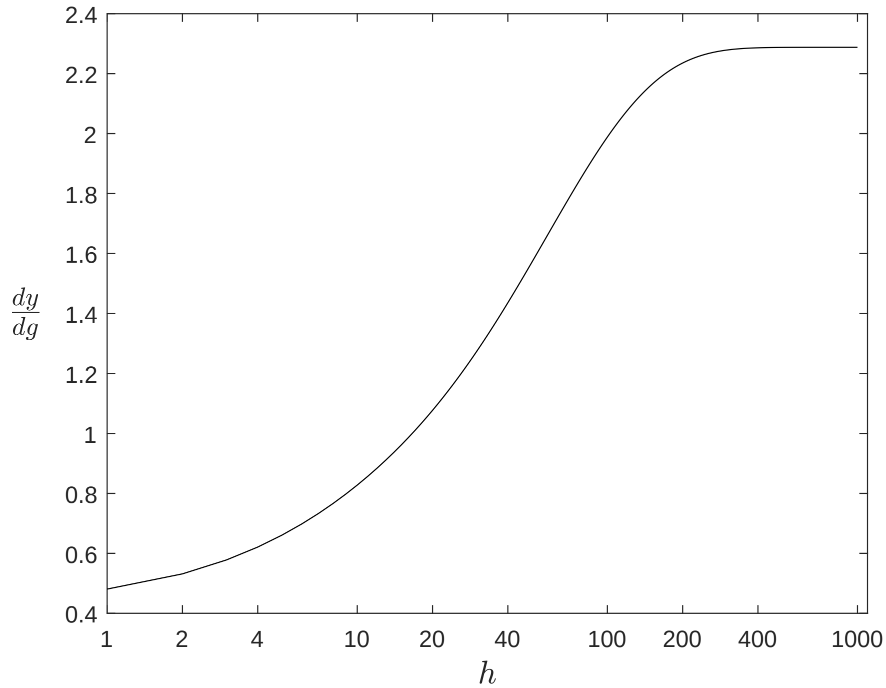

Figure 4: Fiscal Multiplier with Large Government Purchases

Consider a policy under which, when the "crisis" state is entered, the government makes a lump-sum transfer to households, financed by issuing government debt. Thereafter, lump-sum taxes or transfers occur each period of whatever size is needed to maintain a constant level of real public debt. The policy is completely described by a single number, the new level of real public debt $B^* \geq 0$. We assume no change in government purchases to focus on a pure transfer policy.

The policy rule can be written as $B_{t+1} = B^*$ for all $t \geq 0$ where $t = 0$ is the period in which the "crisis" state is entered. It implies that a structural equation for the level of lump-sum taxes collected each period is given by

$$\tau_0 = -\frac{B^*}{1+i_0}$$

$$\begin{aligned}\tau_t &= B_t^*(P_{t-1}/P_t) - \frac{B_{t+1}}{1+i_t} \\ &= B_t^*(P_{t-1}/P_t) - \frac{B^*}{1+i_t}\end{aligned}$$

for all $t \geq 0$.

Monetary policy is assumed to be the same as in Section B: the inflation target is achieved whenever it is consistent with the ZLB; otherwise, the nominal interest rate is set as low as possible. We conjecture that this will mean that during the "crisis" period, the ZLB will bind, so that we will have $i_t = 0$, $P_t \leq P_{t-1}\bar{\Pi}$ for all t; and that during the "normal" period, the inflation target will be achieved, so that $P_t/P_{t-1} = \bar{\Pi}$, $i_t \geq 0$ for all t. We will look for a solution in which the equations are satisfied, and then check that the inequalities are satisfied as well.

In the forward planning by a household with planning horizon h in any period t, it is assumed that all other households also have the same planning horizon h in period t, and a planning horizon $h-j$ in any period $t+j$, for $0 \leq j \leq h$. Hence, the household's plan assumes that, in each period $t+j$ $(0 \leq j \leq h)$, income y_{t+j}^{h-j} will be equal to the amount of spending c_{t+j}^{h-j} that it plans; it follows that the household's plan must involve terminal wealth B_{t+k} equal to the aggregate supply of government debt at that time, namely, B^*. It implies that the F.O.C. for the final period of the plan will be[5]

$$u_c(C_{t+h}^0) = \beta(1 + i_{t+h}^0)v'(B^*) = \beta\frac{1 + i_{t+h}^0}{\bar{\Pi}}u_c(\bar{Y} + \frac{1-\beta}{\bar{\Pi}}B^*)$$

By log-linearization, this becomes

$$c_{t+h}^0 - \varrho_{t+h} = -\sigma\hat{\imath}_{t+h}^0 + (1-\beta)b^*$$

where $b^* = \frac{B^*}{\bar{\Pi}\bar{Y}} \geq 0$ and $log(u_c(C_{t+h}^0)/u_c(\bar{C})) = -\sigma^{-1}(c_{t+h}^0 - \varrho_t)$. Note that the quantity ϱ_t measures the disturbances to the urgency of spending.

Given that $\varrho_t = 0$ at all times in the case that we are analyzing, in the anticipated evolution of the economy, we must have

$$y_{t+h}^0 = -\sigma\hat{\imath}_{t+h}^0 + (1-\beta)b^*$$

at all times. Importantly, the addition of the constant term $(1-\beta)b^*$ is what makes the lump-sum transfer policy different from the outcomes under the policy in Section B. Alternatively, we can rewrite this expression into

$$y_{t+h}^0 = -\sigma\hat{\imath}_{t+h}^0 + y^*$$

where $y^* = (1-\beta)b^* \geq 0$.

Dynamics after reversion to the "normal" state: $(y_t^h, \pi_t^h, \hat{\imath}_t^h)$ are time-invariant, and

[5] The derivation of the F.O.C. can be found in Woodford (2018).

satisfy the recursions as specified in (A.1)-(A.2):

$$\bar{y}^h = \bar{y}^{h-1} - \sigma[\hat{\imath}^h - \bar{\pi}^{h-1}],\ \forall h \geq 1$$

$$\bar{y}^0 = -\sigma\hat{\imath}^0 + y^*$$

$$\bar{\pi}^h = \kappa\bar{y}^h + \beta\bar{\pi}^{h-1},\ \forall h \geq 1$$

$$\bar{\pi}^0 = \kappa\bar{y}^0$$

If we conjecture a solution in which the inflation target is achieved, regardless of the planning horizon h, then $\bar{\pi}^h = 0$ for all $h \geq 0$, and then we must have $\bar{y}^h = 0$ for all $h \geq 0$ as well. It implies

$$\hat{\imath}^0 = \sigma^{-1}y^* \geq 0$$

$$\hat{\imath}^h = 0,\ \forall h \geq 1$$

which are consistent with the ZLB. Since the ZLB is satisfied, the conjecture that there is a solution of this form is confirmed.

Dynamics in the "crisis" state: $(y_t^h, \pi_t^h, \hat{\imath}_t^h)$ are again time-invariant, and satisfy the recursions as specified in (A.1)-(A.2):

$$y^h = [\delta y^{h-1} + (1-\delta)\bar{y}^{h-1}] - \sigma[\hat{\imath}^h + \Delta] + \sigma[\delta\pi^{h-1} + (1-\delta)\bar{\pi}^{h-1}],\ \forall h \geq 1$$

$$y^0 = -\sigma[\hat{\imath}^0 + \Delta] + y^*$$

$$\pi^h = \kappa y^h + \beta[\delta\pi^{h-1} + (1-\delta)\bar{\pi}^{h-1}],\ \forall h \geq 1$$

$$\pi^0 = \kappa y^0$$

Substituting the solution $\bar{\pi}^h = \bar{y}^h = 0$ for all $h \geq 0$, these equations reduce to

$$y^h = \delta y^{h-1} - \sigma[\hat{\imath}^h + \Delta] + \sigma\delta\pi^{h-1},\ \forall h \geq 1$$

$$y^0 = -\sigma[\hat{\imath}^0 + \Delta] + y^*$$

$$\pi^h = \kappa y^h + \beta\delta\pi^{h-1},\ \forall h \geq 1$$

$$\pi^0 = \kappa y^0$$

Consider now whether the ZLB binds for the case of horizon $h = 0$. Achieving $\pi^0 = 0$ would require $y^0 = 0$, and hence $\hat{\imath} = \sigma^{-1}y^* - \Delta$. This is consistent with $\hat{\imath}^0 \geq \underline{\hat{\imath}} = -(r^* + \pi^*)$

if and only if $y^* \geq \sigma\tilde{\Delta} > 0$. Thus, for all $y^* \leq \sigma\tilde{\Delta}$, the ZLB binds for horizon $h = 0$, and

$$y^0 = y^* - \sigma\tilde{\Delta} \leq 0 \tag{D.1}$$

$$\pi^0 = \kappa[y^* - \sigma\tilde{\Delta}] \leq 0 \tag{D.2}$$

If instead $y^* \geq \sigma\tilde{\Delta}$, the inflation target is achieved for horizon $h = 0$, and $\pi^0 = y^0 = 0$, $\hat{\imath}^0 - \underline{\hat{\imath}} = \sigma^{-1}[y^* - \sigma\tilde{\Delta}] \geq 0$. In this case, i.e., $y^* \geq \sigma\tilde{\Delta}$, since $\pi^0 = y^0 = 0$, the recursion reduces to

$$y^h = \delta y^{h-1} - \sigma[\hat{\imath}^h + \Delta] + \sigma\delta\pi^{h-1}, \ \forall h \geq 2$$

$$y^1 = -\sigma[\hat{\imath}^1 + \Delta]$$

$$\pi^h = \kappa y^h + \beta\delta\pi^{h-1}, \ \forall h \geq 2$$

$$\pi^1 = \kappa y^1$$

It is obvious that the ZLB necessarily binds for horizon $h = 1$, and

$$y^1 = -\sigma\tilde{\Delta} < 0 \tag{D.3}$$

$$\pi^1 = -\kappa\sigma\tilde{\Delta} < 0 \tag{D.4}$$

This implies that the ZLB binds even more tightly for horizon $h = 2$, and that y^2, π^2 are even more negative. Hence, the ZLB binds even more tightly for horizon $h = 3$, and so on. One can show that the ZLB binds for all $h \geq 1$, and thus the $\{y^h, \pi^h\}$ for $h \geq 1$ are given by the recursion

$$y^h = \delta y^{h-1} - \sigma\Delta + \sigma\delta\pi^{h-1} \tag{D.5}$$

$$\pi^h = \kappa y^h + \beta\delta\pi^{h-1} \tag{D.6}$$

for all $h \geq 2$, starting from the initial values (D.3)-(D.4) for $h = 1$. Note that conditions (D.3)-(D.4) and (D.5)-(D.6) are exactly the same recursion as equations in (B.3) in Section B, which recursively define the sequences $\{\underline{y}^h, \underline{\pi}^h\}$, except that h corresponds to $h - 1$ in (D.3)-(D.4). Thus, the sequences defined by (D.3)-(D.4) and (D.5)-(D.6) are given by

$$y^h = \underline{y}^{h-1}, \ \pi^h = \underline{\pi}^{h-1}, \ \forall h \geq 1$$

where $\{\underline{y}^h, \underline{\pi}^h\}$ are the sequences studied in Section B. Also, note that this solution is independent of the value of y^*, as long as $y^* \geq \sigma\tilde{\Delta}$.

Suppose instead that $y^* \leq \sigma\tilde{\Delta}$, so that the ZLB binds for horizon $h = 0$, and equations (D.1)-(D.2) hold. In this case, if y^0 and π^0 are negative, this only causes the ZLB to bind even more tightly for horizon $h = 1$, and so on. The above arguments continue to hold, and we again conclude that the ZLB must also bind for all horizons $h \geq 1$. Then, $\{y^h, \pi^h\}$ for all $h \geq 0$ are given by the recursion (D.5)-(D.6) for all $h \geq 1$, not just $h \geq 2$ as above, starting from the initial values (D.1)-(D.2) for $h = 0$.

When $y^* = 0$, this recursion is exactly the same as equations (B.3) in Section B, and has a solution

$$y^h = \underline{y}^h,\ \pi^h = \underline{\pi}^h,\ \forall h \geq 0$$

Furthermore, for any values $0 \leq y^* \leq \sigma\tilde{\Delta}$, the recursive system of equations is linear with a boundary condition that is linear in y^*. It follows that the solutions for each of the $\{y^h, \pi^h\}$ must be linear functions of y^* over the range of $0 \leq y^* \leq \sigma\tilde{\Delta}$. Thus, they must be convex combinations of the solutions for the cases $y^* = 0$ and $y^* = \sigma\tilde{\Delta}$, both of which have already been solved for. We then obtain the general solution

$$y^h = \frac{\sigma\tilde{\Delta} - y^*}{\sigma\tilde{\Delta}}\underline{y}^h + \frac{y^*}{\sigma\tilde{\Delta}}\underline{y}^{h-1},\ \forall h \geq 1 \tag{D.7}$$

$$y^0 = \frac{\sigma\tilde{\Delta} - y^*}{\sigma\tilde{\Delta}}\underline{y}^0 = y^* - \sigma\tilde{\Delta}$$

$$\pi^h = \frac{\sigma\tilde{\Delta} - y^*}{\sigma\tilde{\Delta}}\underline{\pi}^h + \frac{y^*}{\sigma\tilde{\Delta}}\underline{\pi}^{h-1},\ \forall h \geq 1 \tag{D.8}$$

$$\pi^0 = \frac{\sigma\tilde{\Delta} - y^*}{\sigma\tilde{\Delta}}\underline{\pi}^0 = \kappa(y^* - \sigma\tilde{\Delta})$$

Note that since $\underline{y}^h < \underline{y}^{h-1}$, $\underline{\pi}^h < \underline{\pi}^{h-1}$ for all $h \geq 1$, equations (D.1)-(D.2) and (D.7)-(D.8) imply that both y^h and π^h are increasing functions of y^*, and hence of b^*, for all $h \geq 0$ over the range of $0 \leq y^* \leq \sigma\tilde{\Delta}$. Beyond that point, further increases in the size of public debt have no further effect.

One consequence is that, even though a debt financed fiscal transfer to the private sector can reduce the size of output contraction and deflation caused by the "crisis", it does not follow that a large enough "fiscal stimulus" program can eliminate it altogether. The maximum effect of fiscal stimulus – assuming that monetary policy continues to be the inflation targeting regime – is when $y^* = \sigma\tilde{\Delta}$, and in that case, we still have $y^h = \underline{y}^{h-1} < 0$, and $\pi^h = \underline{\pi}^{h-1} < 0$ for all $h \geq 1$. Output and inflation still fall, and indeed the predicted contraction and deflation continue to be quite severe if h is large. Given the calibration in Section A.1, Figure 5 shows the output and inflation in "crisis" state as a function of planning

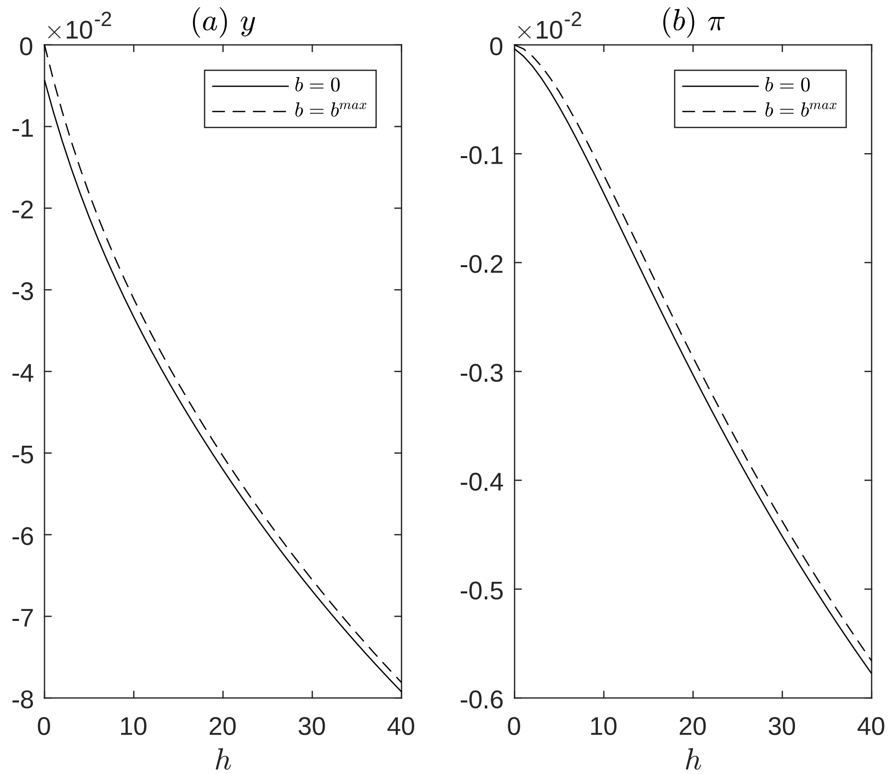

Figure 5: Output and Inflation in "Crisis" State with Different Size of Government Transfers

horizon h with different size of government transfers. In the calibrated example, $b^{max} = 1.44$, i.e., less than 4.5 months' GDP, and any larger b^* than b^{max} does not further reduce output contraction and deflation. Even this magnitude of b^{max} depends on assuming a quarterly planning model. With shorter periods, b^{max} would be even smaller, and approaches zero in the continuous-time limit.

A conclusion is that, in order to completely eliminate the contracting and deflating effects of the "crisis" shock, a transfer policy alone is insufficient: it must be combined with an accomodative monetary policy – that is, the central bank must promise to allow inflation above the target rate π^*, at least during the crisis period. Note that this is not true if the "fiscal stimulus" involves government purchases as studied in Section C, rather than debt-financed transfers alone.

E Lump-sum Transfer Financed by Government Debt with Accomodative Monetary Regime

The limited effect of government transfer studied in Section D is due to the expectation that inflation target would be pursued even during the crisis state, if consistent with the ZLB.

In this section, we consider a combination of fiscal and monetary policies, i.e., combining the government transfer with a commitment to maintain interest rate at the ZLB as long as the "crisis" state persists (instead of strict inflation targeting during the crisis periods), and show that the monetary-fiscal coordination has an effect larger than the sum of the effects of either policy individually.

Consider a fiscal policy similar to that in Section D but with an accomodative monetary policy. More specifically, there is a debt-financed fiscal lump-sum transfer and a permanent increase in the real public debt, parameterized by B^*, the same as the specified fiscal policy in Section D. But, instead of assuming that monetary policy is specified by the inflation target, we consider a different monetary policy: the interest rate is held at the ZLB as long as the "crisis" state continues, even if this involves inflation above the target rate, but as soon as the economy reverts to the "normal" state, monetary policy is again set by the inflation targeting rule. Under this policy, even though for any case with $b^* \leq b^{max}$, the accomodative monetary policy considered will be identical to the policy in Section D, we expect that additional increases in b^* beyond the level b^{max} will provide further stimulus. It would be desirable to explore the effects of larger values of b^* under this policy; in particular, to see how large b^* needs to be, in the calibrated example, in order for there to be no initial decline in output.

Dynamics in the normal state is exactly the same with that in Section D. No further discussion is needed. Dynamics in the "crisis" state, instead, is different under the accomodative monetary policy: $(y_t^h, \pi_t^h, \hat{\imath}_t^h)$ are time-invariant, and satisfy the recursions:

$$y^h = [\delta y^{h-1} + (1-\delta)\bar{y}^{h-1}] - \sigma\tilde{\Delta} + \sigma[\delta\pi^{h-1} + (1-\delta)\bar{\pi}^{h-1}],\ \forall h \geq 1$$

$$y^0 = -\sigma\tilde{\Delta} + y^*$$

$$\pi^h = \kappa y^h + \beta[\delta\pi^{h-1} + (1-\delta)\bar{\pi}^{h-1}],\ \forall h \geq 1$$

$$\pi^0 = \kappa y^0$$

which reduce to

$$y^h = \delta y^{h-1} - \sigma\tilde{\Delta} + \sigma\delta\pi^{h-1},\ \forall h \geq 1$$

$$y^0 = -\sigma\tilde{\Delta} + y^*$$

$$\pi^h = \kappa y^h + \beta\delta\pi^{h-1},\ \forall h \geq 1$$

$$\pi^0 = \kappa y^0$$

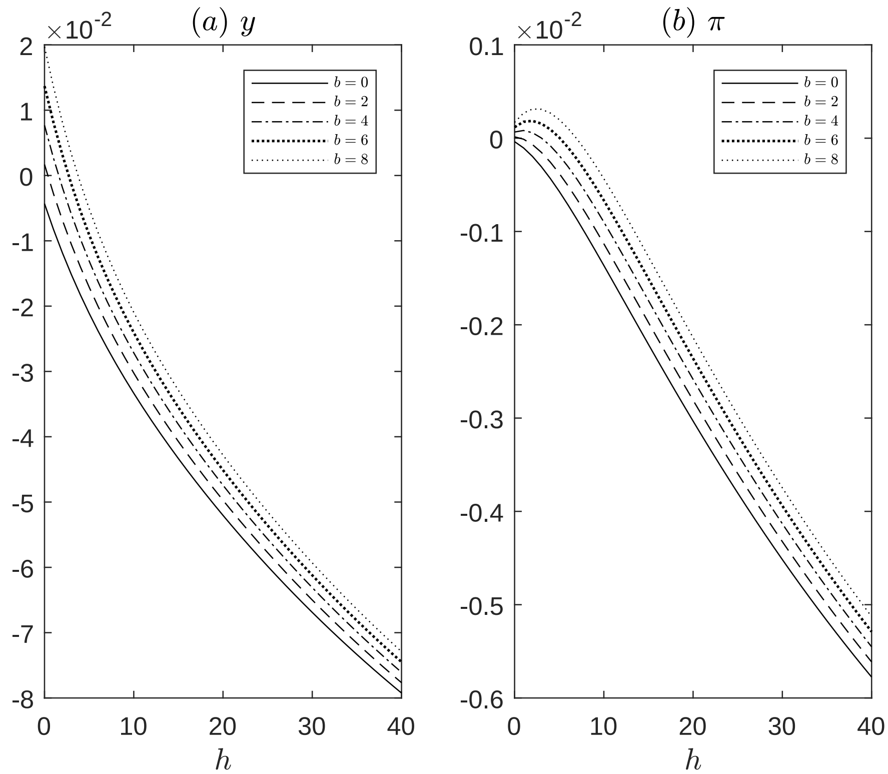

Figure 6: Output and Inflation in “Crisis” State with Coordinated Monetary-fiscal Policy

It is obvious that both y^h and π^h are linear and increasing functions of y^*. Compared with the policy in Section D, the accomodative monetary policy allows the fiscal stimulus to be able to completely eliminate output contraction and deflation as long as the government transfer is large enough. Given the calibration in Section A.1, Figure 6 shows the output and inflation in “crisis” state as a function of planning horizon h with different size of government transfers and accomodative monetary policy. Compared with Figure 5, with an accomodative monetary policy, larger government transfer can improve output contraction and deflation, and even fully eliminate output contraction.

Apart from the fact that it should be a more effective policy than that in Section D, an interesting feature of the coordinated policy is that a “combination” of a change in monetary policy and a change in fiscal policy can accomplish the goal that neither kind of policy can achieve on its own. On the one hand, simply changing monetary policy during the "crisis" state, while policy is expected to be determined by the inflation target outside the “crisis” state, changes nothing; the outcome would continue to be the same as under simply inflation targeting rule in Section B. On the other hand, simply changing fiscal policy as in Section D also accomplishes very little; in fact, as the length of a “period” in the model approaches the continuous-time limit, the effect of fiscal stimulus in Section D, even when b^* is large,

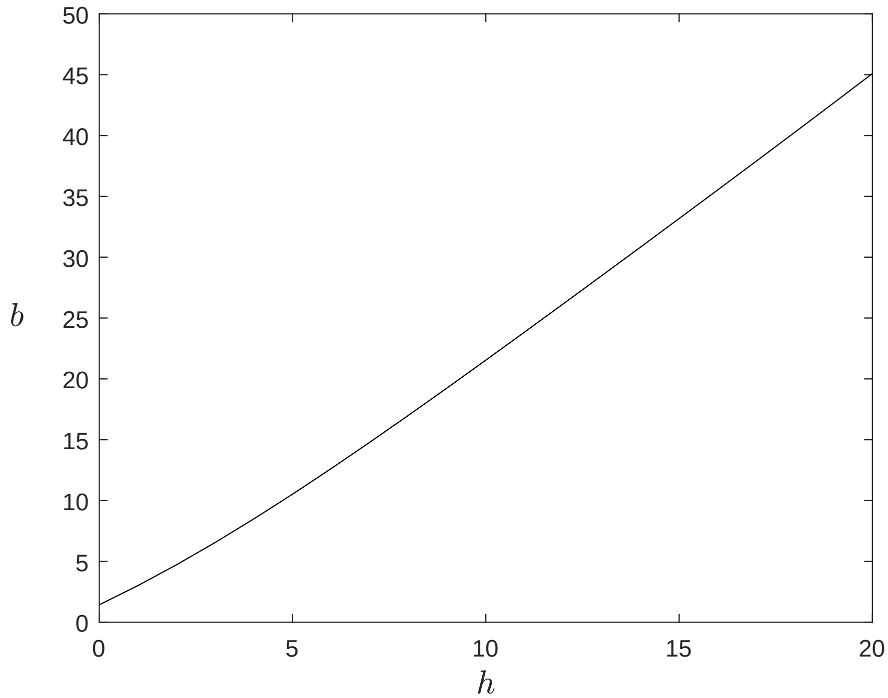

Figure 7: Government Transfers Needed to Fully Eliminate Output Contraction

approaches zero. But "combining" the monetary policy change and the fiscal policy change can instead be quite stimulative. That will be a useful lesson about how the results under the analysis with finite-horizon planning can be quite different from under rational-expectations analysis.

Figure 7 indicates the government transfer needed to fully eliminate output contraction with respect to different planning horizons. Although the coordinated policy could achieve more desirable crisis response, the size of transfer needed is very large. For example, as shown in Figure 6, when $h = 40$, i.e., 10 years, increasing public debt by 200 percent of GDP only raises output during crisis period by less than 1 percent. Another drawback is that the size of transfers needed to prevent output collapse if h is long will be so large as to be highly inflationary if h is short.

Moreover, even if policymakers know the exact distribution of the planning horizons in economy, and can calibrate the size of transfer accordingly, if planning horizons are heterogeneous, then no choice of b^* can avoid distortions induced by the fact that the same policy will be understood as much more expansionary by some households and firms than by others. Other types of commitment to temporary departure from the inflation target may be less prone to such diverse interpretations.

F Temporary Price-level Targeting (TPLT)

The scope of what can be achieved by commitments to (temporarily) looser monetary policy is increased if the central bank can credibly commit to looser monetary policy even after reversion of fundamentals to the "normal" state. In this case, monetary policy commitment can be effective source of stimulus even without any change in fiscal policy. Effects of such "forward guidance" on aggregate demand obviously depend on people's ability to anticipate consequences of different future monetary policy for future economic conditions. Hence, relative to the REE analysis of Eggertsson and Woodford (2003), the predicted effects on output and inflation during the crisis will be weakened with finite planning horizons. Nonetheless, even when horizons are finite if not too short, such policies can still provide an effective form of stimulus.

Moreover, the fact that the effect is weaker in the case of households and firms with shorter horizons is consistent with producing similar responses of spending and price increases on the part of households and firms with heterogeneous horizons, a feature of a desirable stabilization policy as it reduces distortions resulting from differing interpretations of economic outlook.

In this section, we study the effects of a commitment to keep interest rate at the ZLB until price level is restored to trend path with constant inflation rate π^*, i.e., temporary price-level targeting rule. We focus on the case under which the decision makers in the economy do not update their value function, since it can be simply a commitment when a rare financial shock occurs. It captures the idea of "temporary price-level target" suggested by Bernanke (2017).

Consider a policy in which the central bank defines a price level target path $\{P_t^*\}$ that grows deterministically at rate π^*, i.e., $logP_{t+1}^* = logP_t^* + \pi_t^*$ for all t, and achieves this target whenever it is consistent with the ZLB constraint, and sets $\hat{\imath}_t$ as low as possible otherwise. Its policy rule therefore requires that

$$\tilde{P}_t \leq 0 \tag{F.1}$$

at all times, where the price-level gap is defined as $\tilde{p}_t \equiv logP_t - logP_t^*$, and that either $(B.1)$ or $(F.1)$ must hold with equality in each period.

Evolution equation for the price-level gap is given by

$$\tilde{p}_t = \tilde{p}_{t-1} + \pi_t \tag{F.2}$$

where π_t is again the inflation rate in excess of the target rate π_t^*.

The solution is also in the Markovian form: under this policy commitment, the structural equations looking forward from any date t depend only on the value of $\tilde{p}_{t-1}$, which enters $(F.2)$, and the current fundamental state r_t^n either in "normal" or "crisis" state. Thus, once the "normal" state is reached, the solution will be of the form

$$y_t^h = \bar{y}^h(\tilde{p}_{t-1}),\ \pi_t^h = \bar{\pi}^h(\tilde{p}_{t-1})$$

thereafter; while in the "crisis" state, the solution will be of the form

$$y_t^h = \underline{y}^h(\tilde{p}_{t-1}),\ \pi_t^h = \underline{\pi}^h(\tilde{p}_{t-1})$$

Our goal is to compute the functions $\bar{y}^h(\tilde{p}), \bar{\pi}^h(\tilde{p}), \underline{y}^h(\tilde{p}), \underline{\pi}^h(\tilde{p})$ for arbitrary $h \geq 0$ and arbitrary values of $\tilde{p} \leq 0$.

F.1 Temporary Price-level Targeting: Theoretical Derivation

Solutions once the "normal state" is reached: the solution is of the following form. There exists a sequence of critical values $\{\tilde{p}^j\}$ for the price-level gap, which is left to be computed, with the property that

$$\cdots < \tilde{p}^3 < \tilde{p}^2 < \tilde{p}^1 < \tilde{p}^0 < 0 \tag{F.3}$$

and, for any horizon $j \geq 0$, the "price gap" $\tilde{p}^j$ satisfies the property such that (i) if $\tilde{p}_{t-1} \geq \tilde{p}^j$, the price level target is expected to be reached before the end of the planning horizon (i.e., by period $t+j$ or earlier), while (ii) if $\tilde{p}_{t-1} < \tilde{p}^j$, the ZLB is expected to bind over the entire planning horizon, i.e., through period $t+j$.

If $\tilde{p}^j \leq \tilde{p}_{t-1} < \tilde{p}^{j-1}$, then decisions are the same for all horizons $h \geq j$, i.e., for all agents such that $\tilde{p}^h \leq \tilde{p}_{t-1}$: π_t^h and y_t^h depend only on the length of time until the price-level target is expected to be realized, i.e., j periods in the future, not the exact horizon of the agent. But if instead $h < j$, $\pi_t^h = \hat{\pi}^h$ and $y_t^h = \hat{y}^h$ are independent of the value of $\tilde{p}_{t-1}$: π_t^h and y_t^h depend only on the planning horizon, not the size of the current price gap.

Thus, for any price gap $\tilde{p}_{t-1} \leq 0$, there is a horizon $\tau(\tilde{p}_{t-1})$, which is a monotonically decreasing function of $\tilde{p}_{t-1}$ (i.e., as horizon τ is longer, the more negative the price gap is), with the property that the price-level target is expected to be reached at $t+\tau$ by all decision makers with horizons long enough to expect the target to be reached during their planning horizon.

For any planning horizon $h \geq \tau(\tilde{p}_{t-1})$,we have

$$\pi_t^h = \bar{\pi}(\tilde{p}_{t-1}),\ \ y_t^h = \bar{y}(\tilde{p}_{t-1})$$

where the functions $\bar{\pi}(\tilde{p}_{t-1})$, $\bar{y}(\tilde{p}_{t-1})$ are horizon-independent; while for any planning horizon $h < \tau(\tilde{p}_{t-1})$, we have

$$\pi_t^h = \hat{\pi}^h, \; y_t^h = \hat{y}^h$$

where the sequences $\{\hat{\pi}^h, \hat{y}^h\}$ are independent of the price gap. Furthermore, for each $h \geq 0$,

$$\hat{\pi}^h = \bar{\pi}(\tilde{p}^h), \; \hat{y}^h = \bar{y}(\tilde{p}^h)$$

Hence, for any planning horizon h, we have

$$\pi_t^h = \bar{\pi}(\tilde{p}_{t-1}) \text{ if } h \geq \tau(\tilde{p}_{t-1}), \; \pi_t^h = \bar{\pi}(\tilde{p}^h) \text{ otherwise}$$

$$y_t^h = \bar{y}(\tilde{p}_{t-1}) \text{ if } h \geq \tau(\tilde{p}_{t-1}), \; y_t^h = \bar{y}(\tilde{p}^h) \text{ otherwise}$$

or equivalently,

$$\pi_t^h = \bar{\pi}(\tilde{p}_{t-1}) \text{ if } \tilde{p}_{t-1} \geq \tilde{p}^h, \; \pi_t^h = \bar{\pi}(\tilde{p}^h) \text{ if } \tilde{p}_{t-1} \leq \tilde{p}^h$$

$$y_t^h = \bar{y}(\tilde{p}_{t-1}) \text{ if } \tilde{p}_{t-1} \geq \tilde{p}^h, \; y_t^h = \bar{y}(\tilde{p}^h) \text{ if } \tilde{p}_{t-1} \leq \tilde{p}^h$$

By stating in this way, we see that the solution can be completely described by a pair of functions $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$ and the function $\tau(\tilde{p}_{t-1})$, which encodes the values of the sequence $\{\tilde{p}^j\}$.

Another description of the solution can be that there is a function $F(\tilde{p}_{t-1})$ indicating what $\tilde{p}_t$ is expected to be by any agent with a planning horizon $h \geq \tau(\tilde{p}_{t-1})$, i.e., long enough that the price-level target is expected to be achieved within the planning horizon.

When $\tilde{p}_{t-1} \geq \tilde{p}^0$, so that the price-level target is expected to be achieved in the current period, $F(\tilde{p}_{t-1}) = 0$, and the functions $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$ satisfy

$$\begin{aligned}
\bar{y}(\tilde{p}_{t-1}) = \bar{y}(F(\tilde{p}_{t-1})) - \sigma\bar{\imath}(\tilde{p}_{t-1}) + \sigma\bar{\pi}(F(\tilde{p}_{t-1})) \Rightarrow &\qquad \bar{y}(\tilde{p}_{t-1}) = -\sigma\bar{\imath}(\tilde{p}_{t-1}) \\
\bar{\pi}(\tilde{p}_{t-1}) = \kappa\bar{y}(\tilde{p}_{t-1}) + \beta\bar{\pi}(F(\tilde{p}_{t-1})) \Rightarrow &\qquad \bar{\pi}(\tilde{p}_{t-1}) = \kappa\bar{y}(\tilde{p}_{t-1}) \\
\bar{\pi}(\tilde{p}_{t-1}) = F(\tilde{p}_{t-1}) - \tilde{p}_{t-1} \Rightarrow &\qquad \bar{\pi}(\tilde{p}_{t-1}) = -\tilde{p}_{t-1}
\end{aligned}$$

which gives

$$\bar{y}(\tilde{p}_{t-1}) = -\frac{1}{\kappa}\tilde{p}_{t-1}, \; \bar{\pi}(\tilde{p}_{t-1}) = -\tilde{p}_{t-1}, \; \bar{\imath}(\tilde{p}_{t-1}) = \frac{1}{\kappa\sigma}\tilde{p}_{t-1}$$

The boundaries of the range over which this solution applies are the range of values of $\tilde{p}_{t-1}$ for which this solution satisfies the ZLB: $\bar{\imath}(\tilde{p}_{t-1}) \geq -(r^\star + \pi^\star) \Rightarrow \tilde{p}_{t-1} \geq \tilde{p}^0 \equiv -\kappa\sigma(r^\star + \pi^\star)$.

When $\tilde{p}_{t-1} \leq \tilde{p}^0$, the ZLB binds in period t, and the functions $\bar{y}(\tilde{p}_{t-1})$, $\pi(\tilde{p}_{t-1})$ and $F(\tilde{p}_{t-1})$ instead satisfy

$$\bar{y}(\tilde{p}_{t-1}) = \bar{y}(F(\tilde{p}_{t-1})) + \sigma(r^\star + \pi^\star) + \sigma\bar{\pi}(F(\tilde{p}_{t-1})) \tag{F.4a}$$

$$\bar{\pi}(\tilde{p}_{t-1}) = \kappa\bar{y}(\tilde{p}_{t-1}) + \beta\bar{\pi}(F(\tilde{p}_{t-1})) \tag{F.4b}$$

$$\bar{\pi}(\tilde{p}_{t-1}) = F(\tilde{p}_{t-1}) - \tilde{p}_{t-1} \tag{F.4c}$$

The system (F.4a)-(F.4c) can be solved for $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$, $F(\tilde{p}_{t-1})$ if $\tilde{p}_{t-1}$ is in a range such that $\bar{y}(F(\tilde{p}_{t-1}))$ and $\bar{\pi}(F(\tilde{p}_{t-1}))$ are already known as functions of $F(\tilde{p}_{t-1})$. Hence, the solution for $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$ when $\tilde{p}^0 \leq \tilde{p}_{t-1} \leq 0$ allows us to obtain solutions for $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$, $F(\tilde{p}_{t-1})$ for values of $\tilde{p}_{t-1}$ in the interval $\tilde{p}^1 \leq \tilde{p}_{t-1} \leq \tilde{p}^0$, which is the range of values such that (F.4a)-(F.4c) have a solution with $\tilde{p}^0 \leq F(\tilde{p}_{t-1}) \leq 0$.

Note that boundaries of this interval requires that $\tilde{p}^0$ satisfies

$$\bar{y}(\tilde{p}^0) = \bar{y}(0) + \sigma(r^\star + \pi^\star) + \sigma\bar{\pi}(0),\ \bar{\pi}(\tilde{p}^0) = \kappa\bar{y}(\tilde{p}^0) + \beta\bar{\pi}(0),\ \bar{\pi}(\tilde{p}^0) = 0 - \tilde{p}^0$$

and $\tilde{p}^1$ satisfies

$$\bar{y}(\tilde{p}^1) = \bar{y}(\tilde{p}^0) + \sigma(r^\star + \pi^\star) + \sigma\bar{\pi}(\tilde{p}^0),\ \bar{\pi}(\tilde{p}^1) = \kappa\bar{y}(\tilde{p}^1) + \beta\bar{\pi}(\tilde{p}^0),\ \bar{\pi}(\tilde{p}^1) = \tilde{p}^0 - \tilde{p}^1$$

This solution in turn allows us to obtain solutions for $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$, $F(\tilde{p}_{t-1})$ for values in the interval $\tilde{p}^2 \leq \tilde{p}_{t-1} \leq \tilde{p}^1$, which is the range of values such that (F.4a)-(F.4c) have a solution with $\tilde{p}^1 \leq F(\tilde{p}_{t-1}) \leq \tilde{p}^0$.

Here, $\tilde{p}^2$ satisfies

$$\bar{y}(\tilde{p}^2) = \bar{y}(\tilde{p}^1) + \sigma(r^\star + \pi^\star) + \sigma\bar{\pi}(\tilde{p}^1),\ \bar{\pi}(\tilde{p}^2) = \kappa\bar{y}(\tilde{p}^2) + \beta\bar{\pi}(\tilde{p}^1),\ \bar{\pi}(\tilde{p}^2) = \tilde{p}^1 - \tilde{p}^2$$

Similarly, for each progressively lower range of price gaps $\tilde{p}^j \leq \tilde{p}_{t-1} \leq \tilde{p}^{j-1}$, we can solve for $\bar{y}(\tilde{p}_{t-1})$, $\bar{\pi}(\tilde{p}_{t-1})$, $F(\tilde{p}_{t-1})$ and define $\tilde{p}^j$ accordingly. In this way, $F(\tilde{p}_{t-1})$ is a continuous, non-decreasing function defined for all $\tilde{p}_{t-1} \leq 0$; it is constant over the range $\tilde{p}^0 \leq \tilde{p}_{t-1} \leq 0$, where $F(\tilde{p}_{t-1}) = 0$, and otherwise it is monotonically increasing. One can also show that it is a piece-wise linear function, and concave. Meanwhile, by the definition of $\tilde{p}^j$, it is easily verified that the sequence $\{\tilde{p}^j\}_{j=0}^{\infty}$ satisfies the characteristics as conjectured in (F.3). Then,

$\tau(\tilde{p}_{t-1})$ is just the smallest integer such that $F^{\tau}(\tilde{p}_{t-1}) = 0$, and we observe that

$$\tau(\tilde{p}_{t-1}) = j \;\Leftrightarrow\; \tilde{p}^{j} \leq \tilde{p}_{t-1} < \tilde{p}^{j-1}$$

Once we find the function $F(\tilde{p}_{t-1})$, then $\bar{\pi}(\tilde{p}_{t-1}) = F(\tilde{p}_{t-1}) - \tilde{p}_{t-1}$; this will also be a continuous, piece-wise linear function. Also, we have

$$\bar{y}(\tilde{p}_{t-1}) = \frac{1}{\kappa}[\bar{\pi}(\tilde{p}_{t-1}) - \beta\bar{\pi}(F(\tilde{p}_{t-1}))] = \frac{1}{\kappa}[F(\tilde{p}_{t-1}) - \tilde{p}_{t-1} - \beta[F(F(\tilde{p}_{t-1})) - F(\tilde{p}_{t-1})]]$$

which is also a continuous, piece-wise linear function.

Now, suppose that we change the length of the time steps in our discrete-time model, making successive "steps" only a very short period of additional time. In the continuous limit, $\tau(\tilde{p})$ becomes a continuously decreasing function. In this limiting case, we can equivalently describe any given price gap $\tilde{p}$ using the implied length of time $\tau(\tilde{p})$ until the price-level target is expected to be reached, by any agent with a horizon equal to $\tau(\tilde{p})$ or longer. We can rewrite the functions $\bar{y}(\tilde{p})$, $\bar{\pi}(\tilde{p})$ as $\bar{y}(\tau)$, $\bar{\pi}(\tau)$ instead.

In this continuous limit, the system (F.4a)-(F.4c) becomes

$$\frac{d\bar{y}}{d\tau} = \sigma\rho^{\star} + \sigma\bar{\pi}(\tau) \tag{F.5a}$$

$$\frac{d\bar{\pi}}{d\tau} = \gamma[\bar{y}(\tau) - \lambda\bar{\pi}(\tau)] \tag{F.5b}$$

for all $\tau > 0$. Note that $\rho^{\star} > 0$ is the instantaneous nominal interest rate corresponding to the one-period nominal interest rate $r^{\star} + \pi^{\star}$, i.e., $\rho^{\star}\Delta = (r^{\star} + \pi^{\star})$, where Δ is the length of a "period" in the discrete-time model, $\gamma > 0$ is the slope of the continuous time Philips curve relation corresponding to the slope κ in the discrete-time model, i.e., $\gamma = \frac{\kappa}{\Delta^2}$, and $\lambda > 0$ is the slope of the relationship between steady-state inflation and steady-state output implied by the NK Phillips Curve, i.e., $\lambda = \frac{(1-\beta)\Delta}{\kappa}$. The system (F.5a)-(F.5b) is solved from boundary conditions $\bar{\pi}(0) = \bar{y}(0) = 0$.

The system (F.5a)-(F.5b) can be expressed in matrix form:

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & \sigma \\ \gamma & -\lambda\gamma \end{bmatrix} \begin{bmatrix} y + \lambda\rho^{\star} \\ \pi + \rho^{\star} \end{bmatrix} \tag{F.6}$$

where the matrix has two real eigenvalues $\mu_1 < 0 < \mu_2$, i.e., the roots of $\mu^2 + \lambda\gamma\mu - \sigma\gamma = 0$.

The solution consistent with the boundary conditions is then given by

$$\begin{bmatrix} \bar{y}(\tau) \\ \bar{\pi}(\tau) \end{bmatrix} = \begin{bmatrix} -\lambda\rho^\star \\ -\rho^\star \end{bmatrix} + \begin{bmatrix} \frac{\sigma\mu_2}{\mu_1(\mu_2-\mu_1)} \\ \frac{\mu_2}{\mu_2-\mu_1} \end{bmatrix} \rho^\star e^{\mu_1\tau} + \begin{bmatrix} \frac{-\sigma\mu_1}{\mu_2(\mu_2-\mu_1)} \\ \frac{-\mu_1}{\mu_2-\mu_1} \end{bmatrix} \rho^\star e^{\mu_2\tau} \tag{F.7}$$

for all $\tau \geq 0$. We can then integrate the solution for $\bar{\pi}(\tau)$ to obtain

$$\bar{p}(\tau) \equiv -\int_0^\tau \bar{\pi}(s)ds = \rho^\star\tau + \rho^\star \frac{\mu_2}{\mu_1(\mu_2-\mu_1)}[1-e^{\mu_1\tau}] - \rho^\star \frac{\mu_1}{\mu_2(\mu_2-\mu_1)}[1-e^{\mu_2\tau}] \tag{F.8}$$

Note that (F.7) implies that

$$\bar{\pi}(\tau) > -\rho^\star + \rho^\star \frac{\mu_2}{(\mu_2-\mu_1)}(1+\mu_1\tau) - \rho^\star \frac{\mu_1}{(\mu_2-\mu_1)}(1+\mu_1\tau) = 0$$

for all $\tau > 0$, so that $\bar{p}(\tau)$ must be a monotonically decreasing continuous function. Hence, we can invert the function $\bar{p}(\tau)$ to obtain

$$\tau(\tilde{p}) \equiv (\bar{p})^{-1}[\tilde{p}] \geq 0 \tag{F.9}$$

for any $\tilde{p} \leq 0$. Note that, though we cannot give an analytical expression for this function, it can be numerically computed by computing the function $\bar{p}(\tau)$ given by (F.8).

The solution for dynamics in the "normal" state, in the continuous limit, are then given by: for any agent with a horizon h such that $\tilde{p}(t) \geq \bar{p}(h)$, where $\tilde{p}(t)$ is the current existing price-level gap when decision is made and $\bar{p}(h)$ is defined in (F.8), the solution is $\bar{y}^h(t) = \bar{y}(\tau(\tilde{p}(t)))$, $\bar{\pi}^h(t) = \bar{\pi}(\tau(\tilde{p}(t)))$, where the functions $\bar{y}(\tau)$, $\bar{\pi}(\tau)$ are defined in (F.7) and $\tau(\tilde{p})$ is given by (F.9).[6] For any agent with a horizon h such that $\tilde{p}(t) \leq \bar{p}(h)$, the solution is $\bar{y}^h(t) = \bar{y}(h)$, $\bar{\pi}^h(t) = \bar{\pi}(h)$.

Thus, if the economy enters the "normal" state at time T, for an agent with a horizon h such that $\tilde{p}(T) \geq \bar{p}(h)$, the subsequent evolution will be expected to be:

$$y(t) \begin{cases} = \bar{y}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ = 0 & \text{for all } T + \tau(\tilde{p}(T)) \leq t \leq T+h \end{cases}$$

$$\pi(t) \begin{cases} = \bar{\pi}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ = 0 & \text{for all } T + \tau(\tilde{p}(T)) \leq t \leq T+h \end{cases}$$

$$\tilde{p}(t) \begin{cases} = \bar{p}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ = 0 & \text{for all } T + \tau(\tilde{p}(T)) \leq t \leq T+h \end{cases}$$

[6] Note that h is now a continuous length of time instead of a discrete number of periods.

If everyone has the same horizon h, the actual dynamics for $t \geq T$ will be exactly the same as expected. Even though the expected dynamics assumes that everyone's horizon shrinks as time goes forward, and this is not true for the actual dynamics (actually everyone's horizon continues to be h at all times), it continues to be true that $\tilde{p}(t) \geq \bar{p}(h)$ for all $t \geq T$ since $\pi(t) \geq 0$ implies $\tilde{p}(t)$ to be non-decreasing in t. Thus, the solution for the actual dynamics continues to be given by the same formulas as are assumed in people's forward planning.

For an agent with a horizon h such that $\tilde{p}(T) \leq \bar{p}(h)$, the subsequent evolution will instead be expected to be

$$y(t) = \bar{y}(h - (t - T)) \text{ for all } T \leq t \leq T + h$$

$$y(t) = \bar{y}(h - (t - T)) \text{ for all } T \leq t \leq T + h$$

$$\pi(t) = \bar{\pi}(h - (t - T)) \text{ for all } T \leq t \leq T + h$$

$$\tilde{p}(t) = [\tilde{p}(T) - \bar{p}(h)] + \bar{p}(h - (t - T)) \text{ for all } T \leq t \leq T + h$$

But the actual dynamics, if everyone has the same horizon h and $\tilde{p}(T) \leq \bar{p}(h)$, will be given by

$$y(t) = \bar{y}(h),\ \pi(t) = \bar{\pi}(h),\ \tilde{p}(t) = \tilde{p}(T) + \bar{\pi}(h) \cdot (t - T)$$

as long as it continues to be the case that $\tilde{p}(t) \leq \bar{p}(h)$. This latter inequality will hold as long as $t \leq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)}$. Note that at this latter date, $\tilde{p}(t) = \bar{p}(h) \leq 0$, so the ZLB will still be binding, and the solution calculated above will still apply. But after that finite date, we will have $\tilde{p}(t) \geq \bar{p}(h)$, and the solution in the case of $\tilde{p}(T) \geq \bar{p}(h)$ will apply from then on. Thus, the actual dynamics will be given by

$$y(t) = \bar{y}(h),\ \pi(t) = \bar{\pi}(h),\ \tilde{p}(t) = \tilde{p}(T) + \bar{\pi}(h) \cdot (t - T) \text{ for all } T \leq t \leq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)}$$

$$y(t) \begin{cases} = \bar{y}(h - (t - T)) & \text{for all } T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} \leq t \leq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \\ = 0 & \text{for all } t \geq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \end{cases}$$

$$\pi(t) \begin{cases} = \bar{\pi}(h - (t - T)) & \text{for all } T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} \leq t \leq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \\ = 0 & \text{for all } t \geq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \end{cases}$$

$$\tilde{p}(t) \begin{cases} = \bar{p}(h - (t - T)) & \text{for all } T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} \leq t \leq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \\ = 0 & \text{for all } t \geq T + \frac{[\bar{p}(h) - \tilde{p}(T)]}{\bar{\pi}(h)} + h \end{cases}$$

Now, *consider the solution while still in the "crisis"* state: instead of the differential equations specified in the system (F.5a)-(F.5b) that apply in the "normal" state, we have

$$-\frac{dy}{dt} = -\sigma\Delta^{\star} + \sigma\pi + \nu[\bar{y} - y]$$

$$-\frac{d\pi}{dt} = \gamma[y - \lambda\pi] + \nu[\bar{\pi} - \pi]$$

where $\Delta^{\star} > 0$ is the instantaneous nominal rate corresponding to the one-period rate $\tilde{\Delta}$, i.e., $\Delta^{\star} = \frac{\tilde{\Delta}}{\Delta}$ where Δ is the length of one period, and $\nu > 0$ is the continuous arrival rate (Poisson rate) of transitions from the "crisis" state to the "normal" state, i.e., $\nu = \frac{(1-\delta)}{\Delta}$.

Let $y(\tilde{p}, h)$, $\pi(\tilde{p}, h)$ be the solution when the price-level gap is $\tilde{p}$ and decision makers have planning horizon h. Then, in a forward planning exercise, we have

$$\frac{dy}{dt} = \frac{\partial y}{\partial \tilde{p}}\frac{d\tilde{p}}{dt} + \frac{\partial y}{\partial h}\frac{dh}{dt} = \frac{\partial y}{\partial \tilde{p}} \cdot \pi(\tilde{p}, h) - \frac{\partial y}{\partial h}$$

$$\frac{d\pi}{dt} = \frac{\partial \pi}{\partial \tilde{p}}\frac{d\tilde{p}}{dt} + \frac{\partial \pi}{\partial h}\frac{dh}{dt} = \frac{\partial \pi}{\partial \tilde{p}} \cdot \pi(\tilde{p}, h) - \frac{\partial \pi}{\partial h}$$

so that the Euler equations give rise to a system of partial differential equations, i.e.,

$$-\frac{\partial y(\tilde{p}, h)}{\partial \tilde{p}} \cdot \pi(\tilde{p}, h) + \frac{\partial y(\tilde{p}, h)}{\partial h} = -\sigma\Delta^{\star} + \sigma\pi(\tilde{p}, h) + v[\bar{y}(\tilde{p}, h) - y(\tilde{p}, h)] \tag{F.10a}$$

$$-\frac{\partial \pi(\tilde{p}, h)}{\partial \tilde{p}} \cdot \pi(\tilde{p}, h) + \frac{\partial \pi(\tilde{p}, h)}{\partial h} = \gamma[y(\tilde{p}, h) - \lambda\pi(\tilde{p}, h)] + v[\bar{\pi}(\tilde{p}, h) - \pi(\tilde{p}, h)] \tag{F.10b}$$

with boundary conditions $y(\tilde{p}, 0) = \pi(\tilde{p}, 0) = 0$. The functions $\bar{y}(\tilde{p}, h)$, $\bar{\pi}(\tilde{p}, h)$ are the functions given by

$$\bar{y}(\tilde{p}, h) \begin{cases} = \bar{y}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \bar{p}(h) \\ = \bar{y}(h) & \text{if } \tilde{p} \leq \bar{p}(h) \end{cases}$$

$$\bar{\pi}(\tilde{p}, h) \begin{cases} = \bar{\pi}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \bar{p}(h) \\ = \bar{\pi}(h) & \text{if } \tilde{p} \leq \bar{p}(h) \end{cases}$$

which have already been solved in the case of "normal" state.

F.2 Temporary Price-level Targeting: Numerical Methods and Results

Now, we propose a numerical method for approximate solutions of the PDEs (F.10a)-(F.10b): define a discrete grid of values for $(\tilde{p}, h)$ and let (j, k) in the grid corresponding to $\tilde{p} = \bar{p}(j \cdot \epsilon)$ and $h = k \cdot \epsilon$ for some $\epsilon > 0$, where (j, k) are both non-negative integers.[7]

For each integer $j \geq 0$, the goal is to compute sequences of values $\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}$ for progressively higher values of k. Consider the recursive computation, i.e., given the values for $(y(j, k-1), \pi(j, k-1), \tilde{p}(j, k-1))$, we compute

$$y(j, k) = y(j, k-1) - \sigma\Delta^\star\epsilon + \sigma\pi(j, k-1)\epsilon + v\epsilon \cdot [\bar{y}(\tilde{p}(j, k-1), k-1) - y(j, k-1)] \quad \text{(F.11a)}$$

$$\pi(j, k) = \pi(j, k-1) + \gamma\epsilon[y(j, k-1) - \lambda\pi(j, k-1)] + v\epsilon \cdot [\bar{\pi}(\tilde{p}(j, k-1), k-1) - \pi(j, k-1)] \quad \text{(F.11b)}$$

$$\tilde{p}(j, k) = \tilde{p}(j, k-1) - \epsilon\pi(j, k) \quad \text{(F.11c)}$$

The idea is that a sequence $\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}$ represents a possible trajectory $(y(t), \pi(t), \tilde{p}(t))$ along which the economy remains in the "crisis" state, though it is not known in advance that this will be the case. Along this trajectory, $y(t) = y(\tilde{p}(t), h(t))$, $\pi(t) = \pi(\tilde{p}(t), h(t))$ for all t, and $\tilde{p}(t)$ and $h(t)$ evolve according to

$$\frac{d\tilde{p}(t)}{dt} = \pi(t), \ \frac{dh(t)}{dt} = -1$$

The successive values of k index the remaining horizon $h(t)$; the value of j indexes the particular trajectory, which is determined by the terminal values from which one initiates the recursive computation.

For any $j \geq 0$, we start from terminal values $y(j, 0) = \pi(j, 0) = 0$ and some specified value for $\tilde{p}(j, 0)$, then iteratively apply (F.11a)-(F.11c) to compute $\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}$ for progressively higher values of k. Thus, the complete trajectory depends on the value assumed for $\tilde{p}(j, 0) \leq 0$.

In equations (F.11a)-(F.11c), the functions $\bar{y}(\tilde{p}(j, k-1), k-1)$, $\pi(\tilde{p}(j, k-1), k-1)$ are given by:

$$\bar{y}(\tilde{p}, k-1) = \begin{cases} \bar{y}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \bar{p}((k-1) \cdot \epsilon) \\ \bar{y}((k-1) \cdot \epsilon) & \text{if } \tilde{p} \leq \bar{p}((k-1) \cdot \epsilon) \end{cases}$$

[7] The ϵ measures how fine the grid is, and in the numerical calculation, we take $\epsilon = 0.2$. The results are robust by using different values of ϵ.

$$\bar{\pi}(\tilde{p}, k-1) = \begin{cases} \bar{\pi}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \bar{p}((k-1)\cdot\epsilon) \\ \bar{\pi}((k-1)\cdot\epsilon) & \text{if } \tilde{p} \leq \bar{p}((k-1)\cdot\epsilon) \end{cases}$$

We continue iterating the trajectories until either (i) the value $k = K$ is reached, i.e, some pre-specified upper-bound on the range of values of k that are of interest, or (ii) one reaches a value of k at which $\tilde{p}(j,k) \geq 0$.[8] In the latter case, we stop iterating the trajectory at this point, and define

$$k^*(j) \equiv max\{k|\tilde{p}(j,k) < 0\} + 1$$

as the largest value of k for trajectory j.

Now, we characterize the terminal values $\{\tilde{p}(j,0)\}$ for the different values of j. The $\{\tilde{p}(j,0)\}$ is a monotonically decreasing sequence, i.e., $\tilde{p}(j+1,0) < \tilde{p}(j,0) < 0$ for all $j \geq 1$. Furthermore, the sequence of values $\{\tilde{p}(j,0)\}$ that are considered is dense enough, i.e., successive values of $\tilde{p}(j,0)$ should be close enough to one another, to ensure that for every value of k of interest (i.e., a value of k for which we intend to simulate the equilibrium dynamics), there exists some j such that $k^*(j) = k$. In the numerical exercise, given the calibration in Section A.1, we find there existing a $\tilde{p}^* < 0$ such that, for any terminal values $\tilde{p}^* < \tilde{p}(j,0) < 0$, there always exists a $k^*(j)$ as defined before, but for any terminal values $\tilde{p}(j,0) < \tilde{p}^*$, $\tilde{p}(j,k) < 0$ for $\forall k \geq 0$, i.e., $\tilde{p}(j,k)$ becomes negatively exploding starting for some finite k. In other words, in the latter scenario, there does not exist a $k^*(j)$. Figure 8 illustrates the price trajectories with respect to planning horizons, and shows the decreasing sequence of terminal values on y-axis. Nonetheless, as long as we choose the the terminal values $\{\tilde{p}(j,0)\}$ to be smaller than $\tilde{p}^*$ and make it dense enough, we will still be able to find some j such that $k^*(j) = k$ for each $k \leq K$.

Then, for each of the values of k of interest, we define

$$\hat{\jmath}(k) \equiv max\{j|k^\star(h) = k\}$$

Now, we characterize the predicted dynamics in response to a shock. Let the planning horizon k be given (one of the values of k "of interest"). Consider first the dynamics for period $t < N \cdot \epsilon = T$, so that the economy remains in the "crisis state". In any period t of the simulation, we can approximate the value of $\tilde{p}(t)$ by interpolation. We start the simulation in period $t = 0$, and set $\tilde{p}(0) = 0$. The price-level gap is approximated by

$$\tilde{p}(t) = (1-\lambda_t)\tilde{p}(j_t, k) + \lambda_t \tilde{p}(j_t + 1, k)$$

[8] In this numerical calculation, we take $K = 300$.

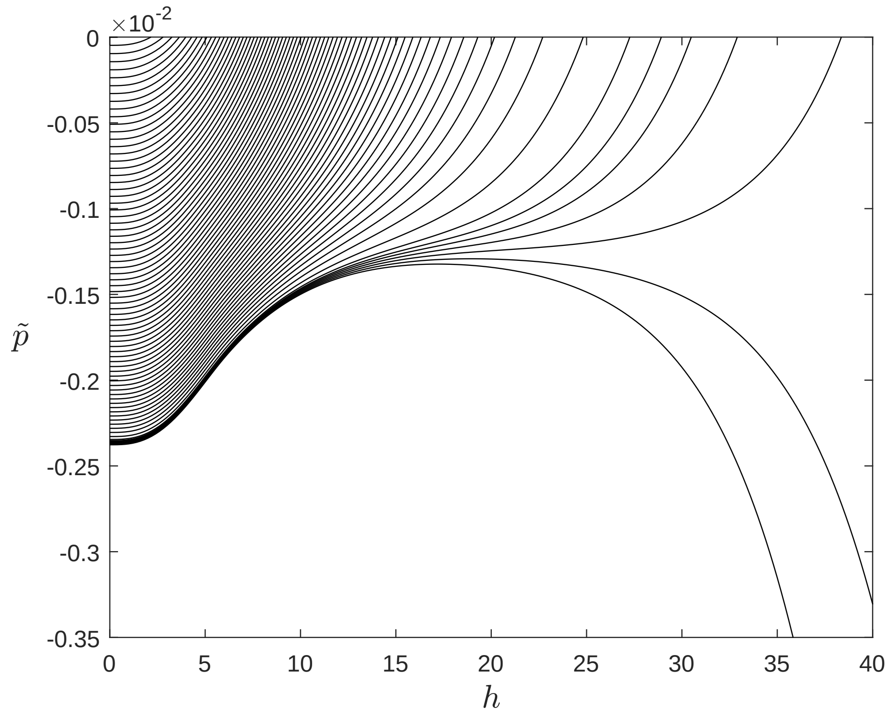

Figure 8: Price-gap Trajectories in "Crisis" State as a Function of Planning Horizon

where

$$j_t \equiv max\{j \geq \hat{\jmath}(k)|\tilde{p}(t) \leq \tilde{p}(j_t, k)\} \tag{F.12a}$$

$$\lambda_t \equiv \frac{\tilde{p}(t) - \tilde{p}(j_t, k)}{\tilde{p}(j_t + 1, k) - \tilde{p}(j_t, k)} \tag{F.12b}$$

Note that, since there exists a $\hat{\jmath}(k)$ such that $\tilde{p}(\hat{\jmath}(k), k) \geq 0$ has been computed, and there also exists at least one higher value of j for which $\tilde{p}(j, k) < 0$ has also been computed, we can necessarily find such a j_t, and both $\tilde{p}(j_t, k)$ and $\tilde{p}(j_t + 1, k)$ will have been computed unless we find that there exist no j such that $\tilde{p}(t) < \tilde{p}(j_t, k)$. The latter problem can be avoided by adding to the first of trajectories j that are computed some additional trajectories starting from lower terminal values $\{\tilde{p}(j, 0)\}$.

We can then approximate the values of $y(\tilde{p}(t), h)$, $\pi(\tilde{p}(t), h)$ by linear interpolation, i.e., for all $0 \leq t \leq N - 1$,

$$y(t) = (1 - \lambda_t)y(j_t, k) + \lambda_t y(j_t + 1, k) \tag{F.13a}$$

$$\pi(t) = (1 - \lambda_t)\pi(j_t, k) + \lambda_t \pi(j_t + 1, k) \tag{F.13b}$$

$$\tilde{p}(t + 1) = \tilde{p}(t) + \pi(t) \cdot \epsilon \tag{F.13c}$$

where the sequences of values $\{y(j, k), \pi(j, k)\}$ for values $j \geq \hat{\jmath}(k)$ have been computed using (F.11a)-(F.11c).

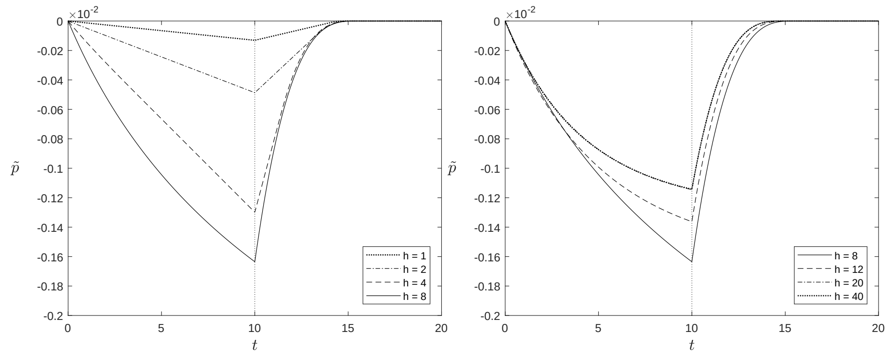

Figure 9: Price-gap Dynamics under Temporary Price-level Targeting

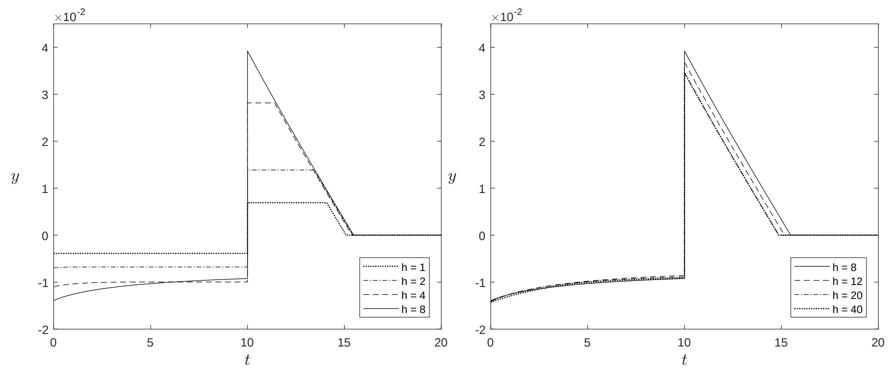

Figure 10: Output Dynamics under Temporary Price-level Targeting

The value at $\tilde{p}(N)$ obtained from (F.13c) when $t = N - 1$ is then the initial condition for the closed-form solution obtained for the period after reversion to the "normal" state.

Under the temporary price-level targeting rule, Figure 9, 10, and 11 show the full dynamics of price, output, and inflation, respectively. In the numerical exercise, given the calibration in Section A.1, we assume that the economy enters "crisis" state at $t = 0$, and reverts to "normal" state at $T = 10$, i.e., 10 quarters after crisis happens.

To compare the results of the temporary price-level targeting with the strict inflation targeting rule as in Section B, Figure 12, 13, and 14 show the comparison of full dynamics for price, output, and inflation, respectively. Though, under the temporary price-level targeting rule, there is an over-shooting for inflation and output after reversion back to the "normal" state, it is much more effective in limiting the effects of the "financial crisis" shock than the standard inflation targeting policy.

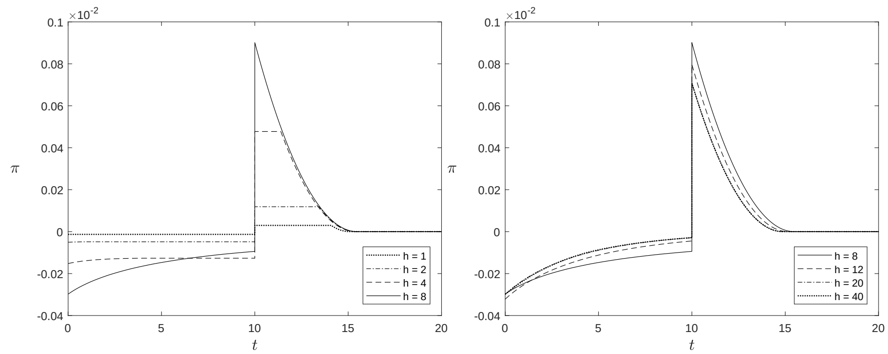

Figure 11: Inflation Dynamics under Temporary Price-level Targeting

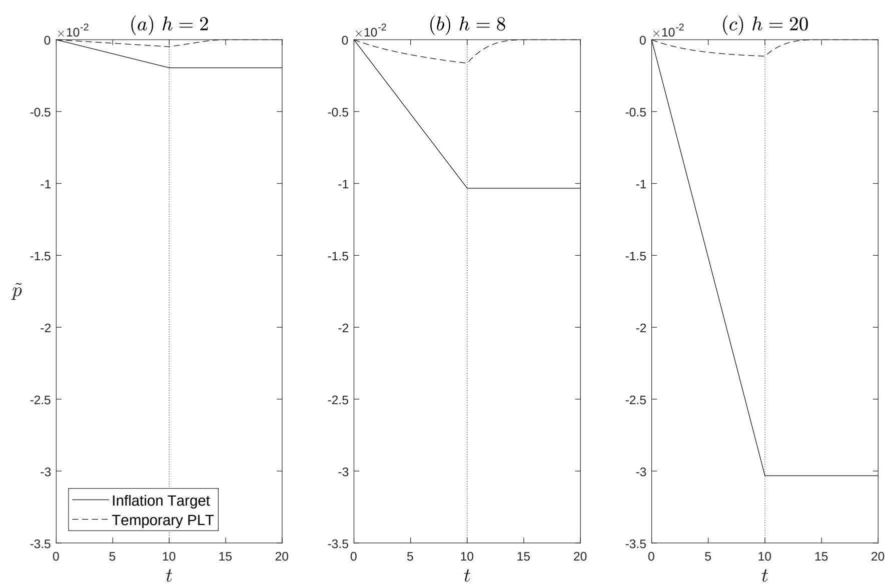

Figure 12: Price-gap Dynamics under Inflation Targeting versus Temporary PLT

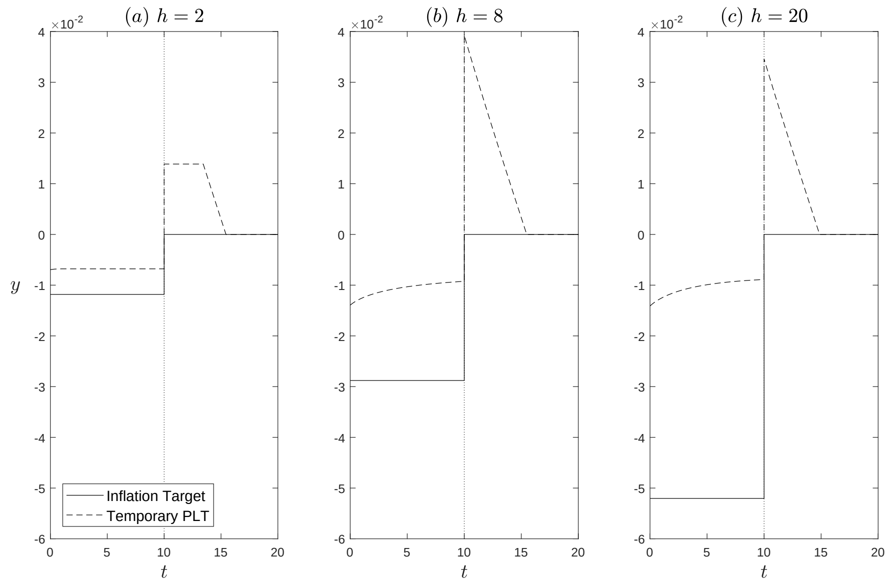

Figure 13: Output Dynamics under Inflation Targeting versus Temporary PLT

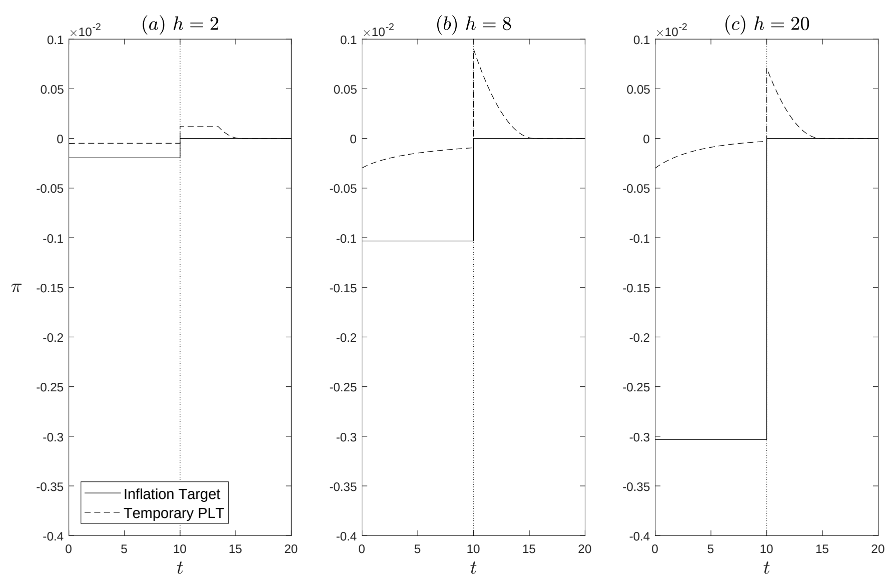

Figure 14: Inflation Dynamics under Inflation Targeting versus Temporary PLT

G Systematic Price-level Targeting (PLT Rule)

In this section, suppose that the price-level target is not simply an ad hoc commitment (form of forward guidance) introduced when the "crisis" shock occurs; instead, it is followed all the time, so that people obtain extensive experience with the dynamics under a price-level target during periods when the "crisis" shock never occurs. In this case, we name the price-level targeting rule as systematic price-level targeting. Then, people should eventually be able to learn the value functions $v(\tilde{p}_{t+k})$, $\tilde{v}(\tilde{p}_{t+k})$ that are correct under such a regime. Under such a regime, v and $\tilde{v}$ are not constants if there are shocks that occasionally cause the central bank to miss the price-level target; instead, v and $\tilde{v}$ depend on the price-level gap at the time that the forward planning is truncated, and the value function is used to evaluate a terminal state.

The two price-level targeting rules, i.e., temporary price-level targeting and systematic price-level targeting, result in different responses during crisis due to the fact that: with finite planning horizon, no matter how credible the temporary commitment might be, pursuit of a different policy systematically outside of crisis periods can allow learning of different value functions by households and firms, and then matter for the behavior during the crisis. In contrast, under REE analysis, one can achieve the same equilibrium response to a financial shock as under a consistently pursued price-level targeting regime simply by committing when such a shock occurs to keep interest rate at the ZLB until the price-level target path is regained.

For the specification of the price-level targeting policy, we adopt similar notations as in Section F.2. The functions $v(\tilde{p})$ and $\tilde{v}(\tilde{p})$ are such that $v = \tilde{v} = 0$ if $\tilde{p} = 0$, i.e., the steady state around which we log-linearate the structural equation is also a stationary equilibrium under the price-level targeting regime, i.e., one in which the price-level target is always achieved, so that $\tilde{p}_t = 0$ at all times.

Dynamics in the "normal" state: once the correct value functions for the normal state have been learned, we have

$$\begin{aligned} \bar{y}(t) &= \bar{y}(\tau(\tilde{p}(t))) \\ \bar{\pi}(t) &= \bar{\pi}(\tau(\tilde{p}(t))) \\ \frac{d\tilde{p}(t)}{dt} &= \bar{\pi}(t) \end{aligned}$$

Starting from initial condition $\tilde{p}(T)$ when the "normal" state is entered at $t = T$, and continuing until date $t^\star$ at which $\tilde{p}(t^\star) = 0$. $\bar{y}(\tau)$, $\bar{\pi}(\tau)$ are again given by equation (F.7) for all $\tau \geq 0$, and $\tau(\tilde{p})$ is the function obtained by inverting the function $\bar{p}(\tau)$ derived in

(F.8). Note that this solution applies regardless of the planning horizon h, and not only when $h \geq \tau(\tilde{p}(t))$ as before; because if a agent has a horizon such that the price-level target is not expected to be reached within the planning horizon, the value function used to evaluate the terminal state is correct, i.e., the same valuation as would be calculated by an agent with a longer planning horizon, which is long enough to see forward to a date at which the price-level target is achieved.

Thus, if the economy enters the "normal" state at date T, regardless of the horizon h, the subsequent evolution is and is expected to be

$$y(t) = \begin{cases} \bar{y}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ 0 & \text{for all } t \geq T + \tau(\tilde{p}(T)) \end{cases}$$

$$\pi(t) = \begin{cases} \bar{\pi}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ 0 & \text{for all } t \geq T + \tau(\tilde{p}(T)) \end{cases}$$

$$\tilde{p}(t) = \begin{cases} \bar{p}(\tau(\tilde{p}(T)) - (t-T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\ 0 & \text{for all } t \geq T + \tau(\tilde{p}(T)) \end{cases}$$

Here, there are no longer "two phases" of the solution for the evolution after date T, i.e., no longer depending on whether $\tilde{p}(t)$ is greater or less than $\bar{p}(h)$; instead, there is only one phase. The solution for the dynamics of $y(t)$, $\pi(t)$, $\tilde{p}(t)$ can be computed in closed form using equations (F.7) and (F.8), once one has determined the value of $\tau(\tilde{p}(T))$.

Now, *consider the dynamics in the "crisis" state*: under the assumption that the value function, which is learned and also used in the "crisis" state, is the one that is correct in the "normal" state (the only state in which people have had prior experiences from which to learn the value function). Then, the trajectory that is anticipated by a finite-horizon planner in the "crisis" state is given by paths $\{y(t), \pi(t), \tilde{p}(t), h(t)\}$, where $h(t)$ is the remaining planning horizon at each point in time t, starting from the time at which the planning takes place. The paths $\{y(t), \pi(t), \tilde{p}(t), h(t)\}$ satisfy a system of differential equations:

$$-\frac{dy}{dt} = -\sigma\Delta^{\star} + \sigma\pi(t) + v[\bar{y}(\tau(\tilde{p}(t))) - y(t)] \tag{G.1a}$$

$$-\frac{d\pi}{dt} = \gamma[y(t) - \lambda\pi(t)] + v[\bar{\pi}(\tau(\tilde{p}(t))) - \pi(t)] \tag{G.1b}$$

$$\frac{d\tilde{p}}{dt} = \pi(t) \tag{G.1c}$$

$$\frac{dh}{dt} = -1 \tag{G.1d}$$

where $\bar{y}(t)$ is the value if there is reversion to the "normal" sate at time t.

The system of (G.1a)-(G.1d) is from time $t = t_0$ (the time at which the planning exercise is undertaken) until $t = t_0 + h_0$ (the actual planning horizon of the agent), starting from initial conditions $\tilde{p}(t_0)$ (given by actual dynamics up to time t_0) and $h(t_0) = h_0$ (the agent's actual planning horizon when the planning exercise is undertaken), and also satisfy terminal conditions

$$y(t_0 + h_0) = \bar{y}(\tau(\tilde{p}(t_0 + h_0))), \ \pi(t_0 + h_0) = \bar{\pi}(\tau(\tilde{p}(t_0 + h_0))) \tag{G.2}$$

Here, the terminal conditions (G.2) reflect the fact that the value functions $v(\tilde{p})$ and $\tilde{v}(\tilde{p})$ that are used when the planning process is terminated at time $t_0 + h_0$ are the ones that would be correct under the dynamics in the "normal" state. That is, these are beliefs that would be correct if it were expected that, at time $t_0 + h_0$, the economy will necessarily revert to the "normal" state if it has not already done so previously. Hence, the terminal beliefs are the same as the beliefs that would be jumped to in the event of a Poisson transition to the "normal" state.

Similar to Section F.2, an anticipated trajectory in the "crisis" state also has to satisfy equations (G.1a)-(G.1d), but instead of terminal conditions (G.2), we impose the terminal conditions

$$y(t_0 + h_0) = 0, \ \pi(t_0 + h_0) = 0$$

regardless of the value of $\tilde{p}(t_0 + h_0)$. In addition to the different terminal conditions, the method used in Section F.2 has a more complex specification of $\bar{y}(\tilde{p}(t), h(t))$; in contrast, equations (G.1a)-(G.1b) under systematic price-level targeting apply only in the case of $h(t) \geq \tau(\tilde{p}(t)) \Leftrightarrow \tilde{p}(t) \geq \tilde{p}(h(t))$ as in Section F.2.

We then numerically solve the system using the same method as in Section F.2. Each trajectory j is associated with a particular possible value $\tau(\tilde{p}(t_0 + h_0)) = \tau_j > 0$. We assume that the $\{\tau_j\}$ are a monotonically increasing sequence. For remaining horizon $k = 0$, from (G.2), we then have

$$y(j, 0) = \bar{y}(\tau_j), \ \pi(j, 0) = \bar{\pi}(\tau_j), \ \tilde{p}(j, 0) = \bar{p}(\tau_j)$$

Starting from these initial values, we then use equations (F.11a)-(F.11c) to recursively calculate $y(j, k)$, $\pi(j, k)$, $\tilde{p}(j, k)$ for progressively higher values of k. The process is continued for values of k up to $k = K$, i.e., the longest horizon of interest for purposes of the simulations, or until one reaches a value of k at which $\tilde{p}(j, k) \geq 0$.

Given a numerical solution for a set of trajectories indexed by j, we can compute

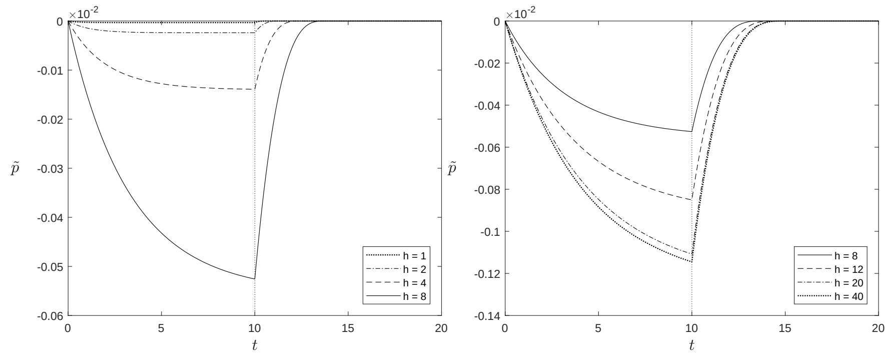

Figure 15: Price-gap Dynamics under Systematic Price-level Targeting (PLT Rule)

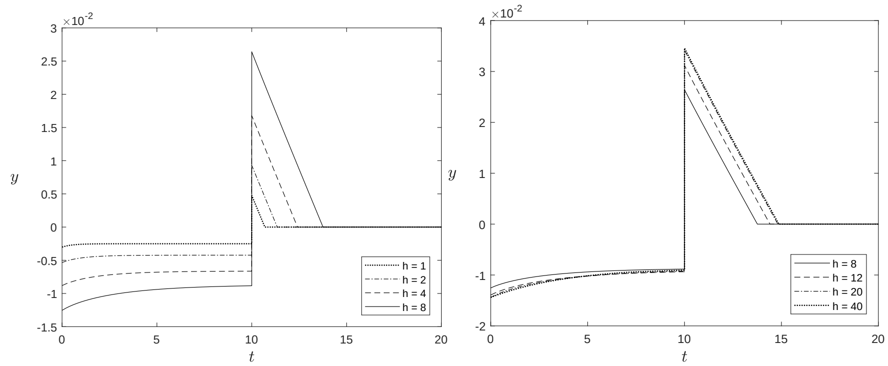

Figure 16: Output Dynamics under Systematic Price-level Targeting (PLT Rule)

simulated paths using the same method as in Section F.2. Under the systematic price-level targeting rule, Figure 15, 16, and 17 show the full dynamics of price, output, and inflation, respectively. In the numerical exercise, as in Section F.2, the economy enters "crisis" state at $t = 0$, and reverts to "normal" state at $T = 10$.

To compare the results of temporary price-level targeting rule with systematic price-level targeting rule, Figure 18, 19, and 20 show the comparison of full dynamics for price, output, and inflation under these two policies, respectively. If the planning horizon is large, e.g., more than five years, i.e., $h \geq 20$, there is not much difference between the two price-level targeting rules. But, if the planning horizon is short for some portion of the people, the systematic price-level targeting rule would improve output and inflation during crisis much better. Thus, following a systematic price-level targeting rule, even when financial frictions are unimportant, can be more effective in limiting the effects of the "financial crisis" shock than a temporary price-level targeting rule introduced only when crisis occurs,

Some might suppose that recognizing limitations on people's ability to correctly anticipate future consequences of a new policy should reduce the benefits from policy commitment, and hence favor a purely discretionary approach to policy. In our analysis, instead, recogniz-

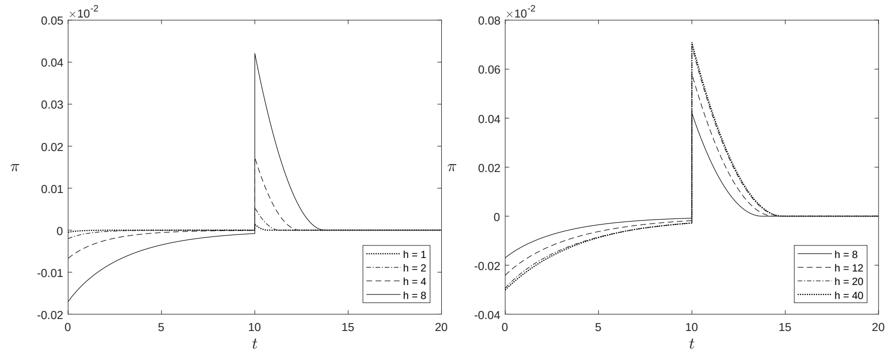

Figure 17: Inflation Dynamics under Systematic Price-level Targeting (PLT Rule)

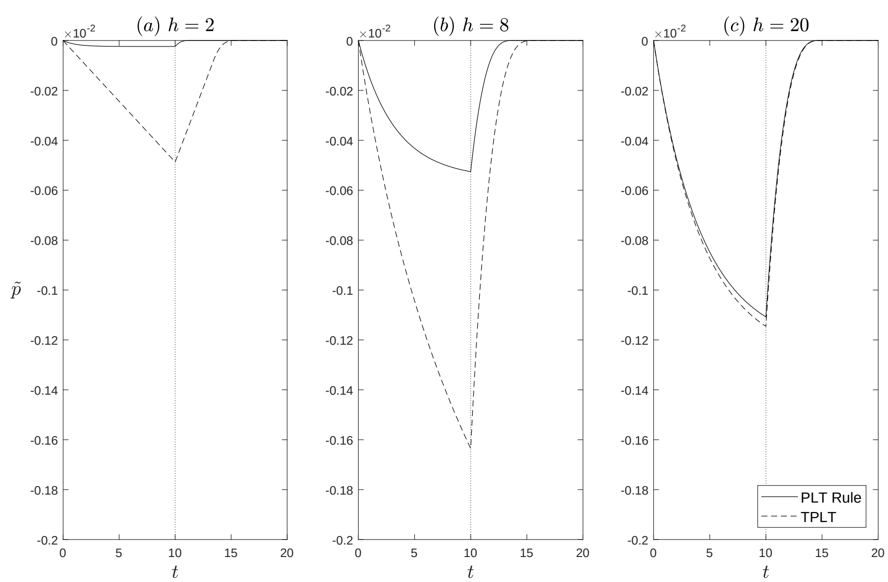

Figure 18: Price-gap Dynamics under Temporary PLT versus PLT Rule

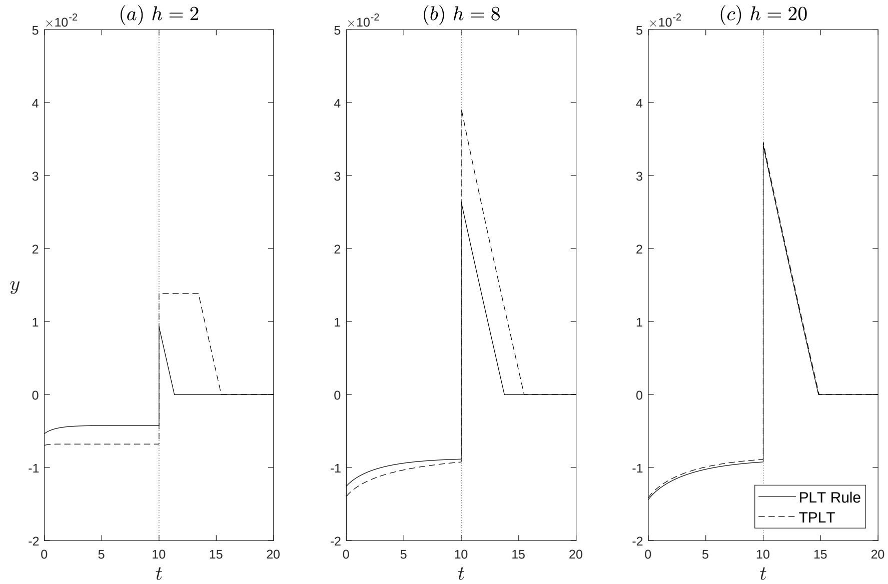

Figure 19: Output Dynamics under Temporary PLT versus PLT Rule

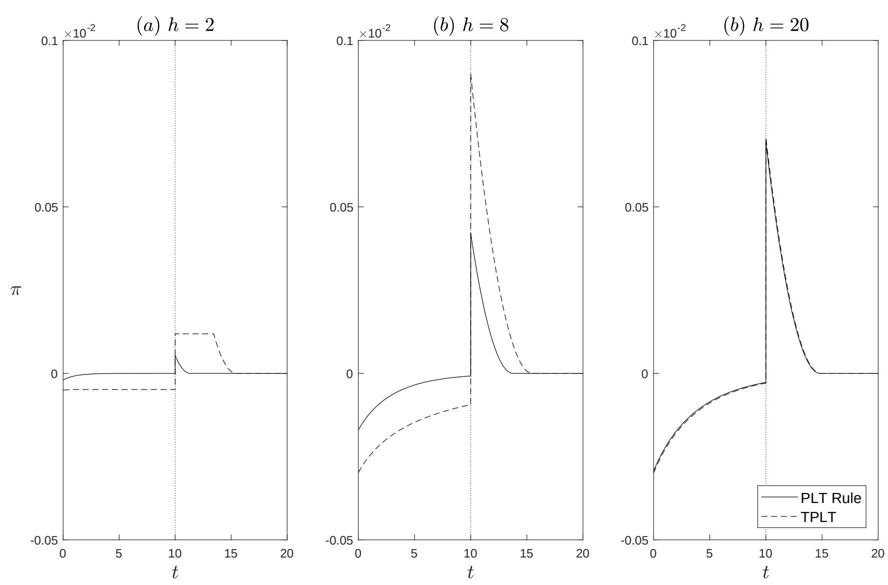

Figure 20: Inflation Dynamics under Temporary PLT versus PLT Rule

ing that planning horizon may not be too long reduces the predicted efficacy of temporary commitments in response to a special situation, and strengthens the case of seeking to design regimes that apply all the time.