Online Appendix

Common Ownership and the Secular Stagnation Hypothesis

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PROOF OF PROPOSITION 1: The objective function of the firm is strictly concave. The second derivative of the objective function with respect to labor is:

$$F_{LL} - 2\omega' - \omega'' \cdot \left(L_j + \lambda L_{-j}\right) < 0$$

since $F_{LL} < 0$ and $-2\omega' - \omega'' \cdot \left(L_j + \lambda L_{-j}\right) < 0$ because we are assuming that labor supply is constant elasticity. The second derivative of the objective function with respect to capital is

$$F_{KK} - 2\rho' - \rho'' \left(K_j + \lambda K_{-j}\right) < 0$$

since $F_{KK} < 0$ and$-2\rho' - \rho'' \left(K_j + \lambda K_{-j}\right) < 0$. The latter inequality follows because $-2\rho' - \rho'' \left(K_j + \lambda K_{-j}\right) = -\rho'(K)\left[2 + \rho''(K)K/\rho'(K)(s_j^K + \lambda(1 - s_j^K))\right]$, where s_j^K is firm j's share of capital and the expression in brackets is positive because $\rho''(K)K/\rho'(K) \geq -1$. To see this, note that $\rho'(K) = \frac{\gamma}{1-\gamma}\frac{E}{E-K}\frac{\rho(K)}{K}$ and $\rho''(K) = \frac{\gamma}{1-\gamma}\frac{\rho(K)}{K^2}\frac{E}{E-K}\left[\frac{K}{E-K} + \frac{\rho'(K)K}{\rho(K)} - 1\right]$. Thus, $\rho''(K)K/\rho'(K) = K/(E-K) + \rho'(K)K/\rho(K) - 1 \geq -1$.

The fact that $F_{LL} \cdot F_{KK} - F_{LK}^2$ is positive (since F is concave) implies that the determinant of the matrix of second derivatives is positive, which is the last condition we needed to establish strict concavity of the objective function. From the first-order conditions, it is then clear that the reaction functions are continuous, and therefore a Nash equilibrium exists.

To prove that there is a unique symmetric equilibrium, we consider the system of FOCs when employment and capital are symmetric across firms, and show that there is a unique solution. From the FOC for labor, we can solve for labor as a function of capital, obtaining:

$$L = \left[\frac{A\alpha}{\chi^{\frac{1}{1-\sigma}}\left(1+\frac{H}{\eta}\right)}\right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} K^{\frac{1-\alpha}{1-\alpha+\frac{1}{\eta}}}.$$

Replacing this in the FOC for capital, we obtain an implicit equation for capital:

$$A(1-\alpha)\left[\frac{A\alpha}{\chi^{\frac{1}{1-\sigma}}\left(1+\frac{H}{\eta}\right)}\right]^{\frac{\alpha}{1-\alpha+\frac{1}{\eta}}} K^{-\frac{\frac{\alpha}{\eta}}{1-\alpha+\frac{1}{\eta}}} - [\rho(K)(1+H/\varepsilon(K)) - (1-\delta)] = 0.$$

The limit when $K \to 0^+$ of this expression is $+\infty$, while the limit when $K \to E^-$ is $-\infty$. The derivative of this expression with respect to K is negative, which implies that there is a unique solution to the equation. The two-equation characterization of the equilibrium obtains directly from imposing symmetry in the FOCs of the firm.

PROOF OF PROPOSITION 2:

(a) We start by noting that the number of firms J and the common ownership parameter ϕ enter the equilibrium equation for capital only through market concentration H. We then use the equilibrium equation for capital to define capital as an implicit function of $H \in (0,1]$:

$$A(1-\alpha)\left[\frac{A\alpha}{\chi^{\frac{1}{1-\sigma}}\left(1+\frac{H}{\eta}\right)}\right]^{\frac{\alpha}{1-\alpha+\frac{1}{\eta}}} K^*(H)^{-\frac{\frac{\alpha}{\eta}}{1-\alpha+\frac{1}{\eta}}} \equiv \rho(K^*(H))(1+H/\varepsilon(K^*(H))) - (1-\delta).$$

Taking log and derivative with respect to $\log H$ yields

$$-\frac{\alpha}{1-\alpha+\frac{1}{\eta}}\left(\frac{\frac{H}{\eta}}{1+\frac{H}{\eta}} + \frac{1}{\eta}\frac{d\log K^*}{d\log H}\right) = \frac{\rho\cdot(1+H/\varepsilon)}{\rho\cdot(1+H/\varepsilon)-(1-\delta)}\left[\frac{1}{\varepsilon}\frac{d\log K^*}{d\log H} + \frac{\frac{H}{\varepsilon}}{1+\frac{H}{\varepsilon}}\left(1+\frac{d\log K^*}{d\log H}\frac{s}{1-s}\right)\right].$$

Solving for $\varepsilon_{KH} \equiv \frac{d\log K^*}{d\log H}$:

$$\varepsilon_{KH} = -\frac{\frac{\alpha}{1-\alpha+\frac{1}{\eta}}\frac{\frac{H}{\eta}}{1+\frac{H}{\eta}} + \frac{\rho\cdot(1+H/\varepsilon)}{\rho\cdot(1+H/\varepsilon)-(1-\delta)}\frac{\frac{H}{\varepsilon}}{1+\frac{H}{\varepsilon}}}{\frac{\alpha}{1-\alpha+\frac{1}{\eta}}\frac{1}{\eta} + \frac{\rho\cdot(1+H/\varepsilon)}{\rho\cdot(1+H/\varepsilon)-(1-\delta)}\left(\frac{1}{\varepsilon}+\frac{\frac{H}{\varepsilon}}{1+\frac{H}{\varepsilon}}\frac{s}{1-s}\right)} < 0.$$

(b) We know that

$$L^* = \left[\frac{A\alpha}{\chi^{\frac{1}{1-\sigma}}\left(1+\frac{H}{\eta}\right)}\right]^{\frac{1}{1-\alpha+\frac{1}{\eta}}} K^{*\,\frac{1-\alpha}{1-\alpha+\frac{1}{\eta}}}.$$

which is decreasing in H and increasing in K. Since H increases when the number of firms decreases or common ownership increases, and K decreases with them, L must decline with both lower J and higher ϕ.

(c), (d), and (e) Since the equilibrium real wage and real interest rates are increasing in L and K, they also must decline when the number of firms decreases or common ownership increases. A lower level of employment and capital also implies lower output.

(f) The labor share of income is $\frac{\omega(L)L}{F(K,L)} = \frac{\alpha}{1+H/\eta}$. A decrease in the number of firms or an increase in the common ownership parameter ϕ increases H and therefore decreases the labor share.

(g) We can obtain:

$$\operatorname{sgn}\left\{\frac{d\log\mu_K^*}{d\log H}\right\} = \operatorname{sgn}\left\{\frac{s}{1-s}\left[\rho(K^*) - \left(\frac{\gamma}{(1-\gamma)s}+1\right)(1-\delta)\right]\varepsilon_{KH} + \rho(K^*) - (1-\delta)\right\}.$$

All else equal, given that $\varepsilon_{KH} < 0$ the expression above is minimized for $\gamma = 0$ for which it becomes:

$$\operatorname{sgn}\left\{\frac{d\log\mu_K^*}{d\log H}\right\} = \operatorname{sgn}\left\{\frac{1-s}{s} + \varepsilon_{KH}\right\}.$$

Thus, a sufficient condition for the real interest rate markup μ_K^* to be increasing in H is that the elasticity of (equilibrium) capital with respect to H be low enough:

$$|\varepsilon_{KH}| < \frac{1-s}{s}.$$

PROOF OF PROPOSITION 3:

As in Azar and Vives (2018), the competitive equilibrium relative price of sector n's good is $\frac{p_n}{P} = \left(\frac{1}{N}\right)^{1/\theta} \left(\frac{c_n}{C}\right)^{-1/\theta}$, where P is the price index $\left(\frac{1}{N}\sum_{n=1}^{N} p_n^{1-\theta}\right)^{1/(1-\theta)}$. The competitive equilibrium relative price of sector n is

$$\psi_n(K,L) = \left(\frac{1}{N}\right)^{1/\theta} \left(\frac{\sum_{j=1}^{J} F(K_{jn}, L_{jn})}{\left[\sum_{m=1}^{N} \left(\frac{1}{N}\right)^{1/\theta} \left(\sum_{j=1}^{J} F(K_{jn}, L_{jn})\right)^{(\theta-1)/\theta}\right]^{\theta/(\theta-1)}}\right)^{-1/\theta}.$$

The derivative with respect to L_{jn} is, as in Azar and Vives (2018):

$$\frac{\partial \psi_n}{\partial L_{jn}} = -\frac{1}{\theta}\psi_n \left(1 - \frac{p_n c_n}{PC}\right) \frac{F_L(K_{jn}, L_{jn})}{c_n} < 0.$$

The derivative with respect to K_{jn} is similar:

$$\frac{\partial \psi_n}{\partial K_{jn}} = -\frac{1}{\theta}\psi_n \left(1 - \frac{p_n c_n}{PC}\right) \frac{F_K(K_{jn}, L_{jn})}{c_n} < 0.$$

Also similarly to Azar and Vives (2018), the derivatives of the relative price in other sectors $m \neq n$ are given by:

$$\frac{\partial \psi_m}{\partial L_{jn}} = \frac{1}{\theta}\psi_n \frac{p_m c_m}{PC} \frac{F_L(K_{jn}, L_{jn})}{c_m} > 0$$

and

$$\frac{\partial \psi_m}{\partial K_{jn}} = \frac{1}{\theta}\psi_n \frac{p_m c_m}{PC} \frac{F_K(K_{jn}, L_{jn})}{c_m} > 0.$$

The first-order condition of firm j with respect to L_{jn} is

$$\psi_n F_L(K_{jn}, L_{jn}) - \omega - \omega' \left(L_{jn} + \lambda_{intra} \sum_{k \neq j} L_{kn} + \lambda_{inter} \sum_{m \neq n} \sum_{k=1}^{J} L_{km} \right) + \frac{\partial \psi_n}{\partial L_{jn}} \left(F(K_{jn}, L_{jn}) + \lambda_{intra} \sum_{k \neq j} F(K_{kn}, L_{kn}) \right) + \lambda_{inter} \sum_{m \neq n} \frac{\partial \psi_m}{\partial L_{jn}} \sum_{k=1}^{J} F(K_{km}, L_{km}) = 0.$$

The first-order condition with respect to K_{jn} is

$$\psi_n F_K(K_{jn}, L_{jn}) - \rho - \rho' \left(K_{jn} + \lambda_{intra} \sum_{k \neq j} K_{kn} + \lambda_{inter} \sum_{m \neq n} \sum_{k=1}^{J} K_{km} \right) + (1 - \delta) + \frac{\partial \psi_n}{\partial K_{jn}} \left(F(K_{jn}, L_{jn}) + \lambda_{intra} \sum_{k \neq j} F(K_{kn}, L_{kn}) \right) + \lambda_{inter} \sum_{m \neq n} \frac{\partial \psi_m}{\partial K_{jn}} \sum_{k=1}^{J} F(K_{km}, L_{km}) = 0.$$

In a symmetric equilibrium, similarly to Azar and Vives (2018), the first-order condition with respect to L_{nj} simplifies to

$$\frac{F_L\left(\frac{K}{J}, \frac{L}{J}\right) - \omega(L)}{\omega(L)} = \frac{\omega'(L)L}{\omega(L)} \left[s^L_{jn} + \lambda_{intra} s^L_{-j,n} + \lambda_{inter} \left(1 - s^L_{jn} - s^L_{-j,n}\right) \right] + \frac{1}{\theta} \left(1 - \frac{1}{N}\right) \frac{F_L(\frac{K}{JN}, \frac{L}{JN})}{\omega(L)} \left[s_{jn} + \lambda_{intra}(1 - s_{jn}) - \lambda_{inter} \right],$$

where $s^L_{jn} \equiv L_{jn}/L$ is the labor market share of firm j in sector n, $s^L_{-j,n} \equiv \sum_{k \neq j} L_{kn}/L$ is the combined labor market share of the other firms in sector n, and $s_{jn} \equiv F(K_{jn}, L_{jn})/c_n$ is the product market share of firm j in sector n.

Analogously, the first-order condition with respect to K_{jn} simplifies to

$$\frac{F_K\left(\frac{K}{JN}, \frac{L}{JN}\right) - \rho(K) + 1 - \delta}{\rho(K) - 1 + \delta} = \frac{\rho'(K)K}{\rho(K) - 1 + \delta} \left[s^K_{jn} + \lambda_{intra} s^K_{-j,n} + \lambda_{inter} \left(1 - s^K_{jn} - s^K_{-j,n}\right) \right] + (1 - \delta) + \frac{1}{\theta} \left(1 - \frac{1}{N}\right) \frac{F_K(\frac{K}{JN}, \frac{L}{JN})}{\rho(K) - 1 + \delta} \left[s_{jn} + \lambda_{intra}(1 - s_{jn}) - \lambda_{inter} \right],$$

where $s^K_{jn} \equiv K_{jn}/K$ is the capital market share of firm j in sector n, $s^K_{-j,n} \equiv \sum_{k \neq j} K_{kn}/L$ is the combined capital market share of the other firms in sector n.

In a symmetric equilibrium the labor market share of firm j in sector n is $\frac{1}{JN}$, its capital market share is also $\frac{1}{JN}$, and its product market share is $\frac{1}{J}$. Since $\frac{\omega'(L)L}{\omega(L)} = \frac{1}{\eta}$, and defining $\mu_L = F_L/\omega - 1$, the first-order condition with respect to L_{jn} can be written as

$$\mu_L^* = \frac{1}{\eta} \underbrace{\left[\frac{1}{JN} + \lambda_{intra}\frac{J-1}{JN} + \lambda_{inter}\frac{N-1}{N}\right]}_{H_{labor}} + \frac{1+\mu_L}{\theta}\left(1-\frac{1}{N}\right)\left[\underbrace{\frac{1}{J} + \lambda_{intra}\frac{J-1}{J}}_{H_{product}} - \lambda_{inter}\right].$$

Similarly, since $\frac{\rho'(K)K}{\rho(K)-1+\delta} = \frac{1}{\varepsilon(K)}\frac{1}{1-\frac{1-\delta}{\rho(K)}}$, and defining $\mu_K = F_K/(\rho - 1 + \delta) - 1$, the first-order condition with respect to capital can be written as

$$\mu_K^* = \frac{1}{\varepsilon(K)\left(1-\frac{1-\delta}{\rho(K)}\right)} \underbrace{\left[\frac{1}{JN} + \lambda_{intra}\frac{J-1}{JN} + \lambda_{inter}\frac{N-1}{N}\right]}_{H_{capital}} + \frac{1+\mu_K}{\theta}\left(1-\frac{1}{N}\right)\left[\underbrace{\frac{1}{J} + \lambda_{intra}\frac{J-1}{J}}_{H_{product}} - \lambda_{inter}\right].$$

Solving for $1+\mu_L^*$ and $1+\mu_K^*$, we obtain

$$1+\mu_L^* = \frac{1+H_{labor}/\eta}{1-\left(H_{product}-\lambda_{inter}\right)(1-1/N)/\theta}$$

$$1+\mu_K^* = \frac{1+H_{capital}/(\varepsilon(K)(1-(1-\delta)/\rho(K)))}{1-\left(H_{product}-\lambda_{inter}\right)(1-1/N)/\theta},$$

which are the expressions for the markdowns in the proposition.